M 26089

Reg. No. : ....

Name : ....

Third Semester M.A./M.Sc./M.Com. Degree (Reg./Sup./Imp.) Examination, November 2014

MATHEMATICS

(2007 Admn. Onwards)

Paper – XI : Differential Geometry

Time: 3 Hours Max. Marks : 60

PART – A

Answer **any four** questions. **Each** question carries **3** marks.

1. Define a vector field on a set $U \subset \mathbb{R}^{n+1}$. Sketch the vector field $\mathbb{X}(p) = (p, X(p))$ where $X(p) = (0, 1)$ on $\mathbb{R}^2$.
2. Let $f : U \to \mathbb{R}$ be a smooth function on U, U open in $\mathbb{R}^n$. Then prove that the graph of f is an n-surface in $\mathbb{R}^{n+1}$.
3. Prove that geodesics have constant speed.
4. Find the velocity, the acceleration, and the speed of the parameterized curve $\alpha(t) = (\cos t, \sin t)$.
5. If $\mathbb{X}$ and Y are smooth vector fields tangent to S along a parameterized curve $\alpha : I \to S$, then prove that $(\mathbb{X} \cdot Y)' = X' \cdot Y + \mathbb{X} \cdot Y'$
6. Define the length of a parameterized curve in $\mathbb{R}^{n+1}$. Also show that it is invariant under reparameterization. **(4×3=12)**

PART – B

Answer **any four** questions without omitting **any** Unit. **Each** question carries 12 marks.

UNIT – I

7. a) Let X be a smooth vector field on an open set $U \subset \mathbb{R}^{n+1}$ and let $p \in U$. Then show that there is a maximal integral curve of $\mathbb{X}$ thought p.

   b) Sketch the vector field $\mathbb{X}(p) = (p, X(p))$ where $X(p) = -p$ on $\mathbb{R}^2$. Also find the integral curve of $\mathbb{X}$ through $p = (1, 1)$.

P.T.O.

8. a) Let U be an open set in $\mathbb{R}^{n+1}$ and let $f = U \to \mathbb{R}$ be smooth. Let $p \in U$ be a regular point of f, and let $c = f(p)$. Then prove that the set of all vectors tangent to $f^{-1}(c)$ at p is equal to $[\nabla f(p)]^{\perp}$.

   b) Let $f : U \to \mathbb{R}$ be a smooth function and let $\alpha : I \to U$ be an integral curve of ∇f. Show that $\frac{d}{dt}(f \circ \alpha)(t) = \| \nabla f(\alpha(t)) \|^2$ for all $t \in I$.

9. a) State and prove the Lagrange multiplier theorem.

   b) Show that the maximum and minimum values of the function $g(x_1, x_2) = a x_1^2 + 2bx_1x_2 + c x_2^2$ where $a, b, c \in \mathbb{R}$ on the unit circle $x_1^2 + x_2^2 = 1$ are the eigen values of the matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$.

UNIT – II

10. a) Show that if $\alpha : I \to \mathbb{R}^{n+1}$ is a parameterized curve with constant speed, then $\ddot{\alpha}(t) \perp \dot{\alpha}(t)$ for all $t \in I$.

    b) Show that for each $a, b, c, d \in \mathbb{R}$, the parameterized curve $\alpha(t) = (\cos(at + b), \sin(at + b), (t + d))$ is a geodesic in the cylinder $x_1^2 + x_2^2 = 1$ in $\mathbb{R}^3$.

    c) State the properties of Levi-Civita parallelism.

11. a) Let S be an n-surface in $\mathbb{R}^{n+1}$, let $p, q \in S$, and let α be a piecewise smooth parameterized curve from p to q. Then prove that the parallel transport $P_\alpha : S_p \to S_q$ along α is a vector space isomorphism which preserves dot products.

    b) Define the Weingarten map of an oriented n-surface S in $\mathbb{R}^{n+1}$. Compute the Weingarten map for the circular cylinder $x_2^2 + x_3^2 = a^2$ in $\mathbb{R}^3$, $a \neq 0$ by choosing an orientation.

12. a) Prove that the Weingarten map L_p is self-adjoint.

    b) Prove that local parameterizations of plane curves are unique upto reparameterization.

UNIT – III

13. a) Let C be a connected oriented plane curve and let $\beta : I \to C$ be a unit speed global parameterization of C. Then prove that β is either one to one or periodic.

    b) Prove that for each 1-form w on U (U open in $\mathbb{R}^{n+1}$) there exist unique functions $f_i : U \to \mathbb{R}$ ($i \in \{1, 2, ..., n + 1\}$) such that $w = \sum_{i=1}^{n+1} f_i \, dx_i$.

14. a) Let V be a finite dimensional vector space with dot product and let $L : V \to V$ be a self-adjoint linear transformation on V. Then prove that there exists an orthonormal basis for V consisting of eigen vectors of L.

    b) Find the Gaussian curvature of the ellipsoid $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1$ (a, b, c all $\neq 0$) oriented by its outward normal.

15. a) Let $\varphi : U \to \mathbb{R}^{n+1}$ be a parameterized n-surface in $\mathbb{R}^{n+1}$ and let $p \in U$. Then prove that there exists an open set $U_1 \subset U$ about p such that $\varphi(U_1)$ is an n-surface in $\mathbb{R}^{n+1}$.

    b) State and prove the inverse function theorem for n-surfaces. **(4x12=48)**