Reg. No. : ....

Name : ....

M 26090

Third Semester M.A./M.Sc./M.Com. Degree (Reg./Sup./Imp.)
Examination, November 2014
(2007 Admn. Onwards)
MATHEMATICS
Paper – XII : Complex Analysis

Time: 3 Hours Max. Marks : 60

PART – A

Answer **any four** questions. **Each** question carries **3** marks.

1. Compute $\int_{|z|=2} \frac{dz}{z^2+1}$ by decomposition of the integrand into partial fractions.

2. How many roots of the equation $z^4 - 6z + 3 = 0$ have their modulus between 1 and 2 ?

3. Define a harmonic function. If u is harmonic in a region Ω show that $f(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ is analytic in Ω.

4. Show that the series $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ converges for Re z > 1 and represent the derivative in series form.

5. Develop the function $f(z) = (1+z)^{\mu}$ is Taylor series about the origin, choosing a branch of f(z) which equals 1 at the origin.

6. Prove that an elliptic function without poles is constant. **(4×3=12)**

PART – B

Answer **any four** questions without omitting **any** Unit. **Each** question carries **12** marks.

UNIT – I

7. a) State and prove Cauchy's theorem for a rectangle.

   b) Prove that a function which is analytic and bounded in the whole plane must reduce to a constant.

8. a) Prove that an analytic function comes arbitrarily close to any complex value in every neighborhood of an essential singularity.

   b) If f(z) is analytic and nonconstant in a region Ω, prove that $|f(z)|$ has no maximum in Ω.

   c) State and prove lemma of Schwarz.

9. a) State and prove the residue theorem.

   b) Evaluate $\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx$ by the method of residues.

UNIT – II

10. a) Prove that the arithmetic mean of a harmonic function over concentric circles $|z| = r$ is a linear function of log r, $\frac{1}{2\pi} \int_{|z|=r} u \, d\theta = \alpha \log r + \beta$.

    b) Prove that the function $P_U(z)$ is harmonic for $|z| < 1$ and $\lim_{z \to e^{i\theta_0}} P_u(z) = U(\theta_0)$.

11. a) State and prove Mittag-Leffler theorem.

    b) Comparing the coefficients in the Laurent developments $\cot \pi z$ and of its expression as a sum of partial fractions find the value of $\sum_{1}^{\infty} \frac{1}{n^2}$.

12. a) Prove that the infinite product $\prod_{1}^{\infty} (1 + a_n)$ with $1 + a_n \neq 0$ converges simultaneously with the series $\sum_{1}^{\infty} \log(1 + a_n)$ whose terms represent the principal branch of the logarithm.

    b) Derive the factorization $\sin \pi z = \pi z \prod_{1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$.

UNIT – III

13. a) Prove that the ζ–function can be extended to a monomorphic function in the whole plane with a simple pole at s =1 with the residue 1.

    b) Prove the functional equation :

    $$\zeta(s) = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1 - s) \zeta(1 - s).$$

14. a) Prove that a discrete module consists either zero alone, of the integral multiples nw of a single complex number $w \neq 0$, or all linear combinations $n_1 w_1 + n_2 w_2$ with integral coefficients of two numbers with non real ratio $\frac{w_2}{w_1}$.

    b) Prove that any two bases of the same module are connected by a unimodular transformation.

15. a) Define the Weirstrass's function $\zeta(z)$ and prove that

    $\zeta(z + w_1) = \zeta(z) + \eta_1$; $\zeta(z + w_2) = \zeta(z) + \eta_2$

    where η_1 and η_2 are constants. Also prove that the quantities η_1, η_2, w_1, w_2 are connected by the relation $\eta_1 w_2 - \eta_2 w_1 = 2\pi i$.

    b) With usual notations prove that the Weirstrass $\wp$ - function satisfies the differential equation $\wp^1(z)^2 = 4\wp(z)^3 = g_2\wp(z) - g_3$. **(4×12=48)**
