M 26091

Reg. No. : ....

Name : ....

Third Semester M.A./M.Sc./M.Com. Degree (Reg./Sup./Imp.) Examination, November 2014

MATHEMATICS

(2007 Admn. Onwards)

Paper – XIII : Functional Analysis – I

Time : 3 Hours Max. Marks : 60

Instructions : *Answer **any four** questions from Part – **A**. **Each** question carries 3 marks.*

*Answer **any four** questions from Part – **B** without omitting **any** Unit. **Each** question carries 12 marks.*

PART – A

1. Let d and d′ be two metrics on a set X. If d is stronger than d′ then prove that whenever a sequence $x_n \to x$ in X with respect d, the sequence $x_n \to x$ in X w.v. to d′.

2. Prove that the space $L^{\infty}([a, b])$ is not separable.

3. Let M= (k_{ij}) be an m × n matrix with scalar entries. Suppose $X = \mathbb{K}^n$ with $\| \cdot \|_1$ and $Y = \mathbb{K}^m$ with $\| \cdot \|_1$, prove that the map $M : X \to Y$ defined by $Mx(i) = \sum_{j=1}^{n} k_{ij} x(j)$, i = 1, 2...m is linear and its norm is $\underset{j=1,2-n}{\leq \max} \sum_{i=1}^{m} | k_{ij} |$

4. Let $X = \mathbb{K}^2$ with the norm $\| \cdot \|_{\infty}$. Suppose $Y = \{(x(1), x(2)) : x(2) = 0\}$ If $g(x(1), x(2)) = x(1), (x(1), x(2)) \in X$ then prove that $g \in Y'$ and g has only one Hahn Banach extension.

P.T.O.

5. If T is a metric space, show that $C_0(T)$ is a Banach space with respect to the supremum norm.

6. Let X and Y be normed spaces and $F : X \to Y$ be linear. If $g \circ F$ is continuous for every $g \in Y'$ prove that F is continuous. **(4×3=12)**

PART – B

Unit – I

7. a) State and prove Korovkin's theorem and deduce that the set of polynomials in one variable is dense in c([a, b]) with the sup metric.

   b) For $E \subset \mathbb{R}$, let C_E denote the characteristic function of E. Prove that if E is measurable then C_E is measurable.

8. a) Prove that if $x \in L'[-\pi, \pi]$ then $\hat{x}(n) \to 0$ as $n \to \pm\infty$.

   b) Show that any norm on the scalar field $\mathbb{K}$ (by seeing it as a vector space over $\mathbb{K}$) is a positive scalar multiple of the absolute value function.

9. a) Show by examples the spaces $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ are not comparable under inclusion.

   b) State and prove Riesz lemma for normed spaces.

Unit – II

10. a) Show by an example that not all linear functionals on an infinite dimensional normed space are continuous.

    b) Let X be a normed space over $\mathbb{K}$ and f be a non zero linear functional on X. If E is an open subset of X then prove that f(E) is an open subset of $\mathbb{K}$.

    c) Let E be a nonempty convex subset of a normed space X over $\mathbb{K}$. If $a \in X \setminus \overline{E}$, prove that there are $f \in X'$ and $f \in \mathbb{R}$ such that $\text{Re} f(x) \le t < \text{Re} f(a)$ for all $x \in \overline{E}$.

11. a) Let X be a normed space over $\mathbb{K}$, Y be a subspace of X and $g \in Y'$. Prove that there exists an $f \in X'$ such that $f/Y = g$ and $\| f \| = \| g \|$.

    b) Define an innerproduct space and show that among all the normed spaces $L^p([0, 1])$, $1 \le p \le \infty$, only the space $L^2([0, 1])$ is an inner product space.

12. a) Let Y be a subspace of X and $a \in X \setminus \overline{Y}$. Then prove that there exists some $f \in X'$ such that $f(a) = \| a \|$ and $\| f \| = 1$.

    b) State and prove Schwarz inequality for element of an innerproduct space.

    c) Let X be an innerproduct space and $\{x_1, \ldots x_n\}$ be an orthonormal set in X. For any scalars $k_1, k_2 \ldots k_n$ prove that $\| k_1 x_1 + k_2 x_2 + \ldots + k_n x_n \|^2 = | k_1 |^2 + \ldots + | k_n |^2$.

Unit – III

13. a) Show that a normed space X is Banach if and only if every absolutely summable series of elements of X is summable in X.

    b) Let X be a Banach space, Y be a normed space and $\mathfrak{F}$ be a subset of BL (X, Y) such that for each $x \in X$, the set $\{F(x) = F \in \mathfrak{F}\}$ is bounded in Y. Prove that $\mathfrak{F}$ is uniformly bounded on E.

14. a) Let X and Y be metric spaces. Prove that every continuous map $F : X \to Y$ is closed. What about the converse ? Justify your claim.

    b) Let X be a normed space and p be a projection. Then prove that a necessary and sufficient condition for closedness of p is the closedness of the subspaces R(p) and Z(p) in X.

15. a) Let X be a normed space and E be a subset of X prove that E is bounded in X if and only if f(E) is bounded in $\mathbb{K}$ for every $f \in X'$.

    b) Let X and Y be Banach spaces. If F is a bijective linear bounded map from X and Y, prove that $F^{-1} : Y \to X$ is bounded. What about this result when X and Y fail to be Banach ? Justify your claim. **(4×12=48)**

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