b) Let V be a finite dimensional vector space and let T be a linear operator on V and let α be any nonzero vector in V and let p_α be the T-annihilator of α. Then prove the following

i) the degree of p_α is equal to the dimension of the cyclic subspace $Z(\alpha;T)$.

ii) If the degree of p_α is k, then the vectors $\alpha, T\alpha, T^2\alpha, \ldots, T^{k-1}\alpha$ form a basis for $Z(\alpha;T)$.

iii) If U is the linear operator on $Z(\alpha; T)$ induced by T, then the minimal polynomial for U is p_α.

14. a) Let T be a linear operator on the finite dimensional vector space V over the field F. Suppose that the minimal polynomial for T decomposes over F into a product of linear polynomials. Then prove that there is a diagonalizable operator D on V and a nilpotent operator N on V such that

i) $T = D + N$,

ii) $DN = ND$.

Also shows that the diagonalizable operator D and the nilpotent operator N are uniquely determined by (i) and (ii) and each of them is a polynomial in T.

b) If A is the companion matrix of a monic polynomial p, then prove that p is both the minimal and the characteristic polynomial of A.

15. a) Define orthonormal set in an inner product space and give an example.

b) Show that an orthogonal set of nonzero vectors is linearly independent.

c) Let W be a finite-dimensional subspace of an inner product space V and let E be the orthogonal projection of V on W. Then prove that E is an idem-potent linear transformation of V onto W, $W^\perp$ is the null space of E, and $V = W \oplus W^\perp$.

Reg. No. : ....

Name : ....

First Semester M.Sc. Degree (CBSS – Reg./Suppl. (Including Mercy Chance)/Imp.) Examination, October 2020

(2017 Admission Onwards)

MATHEMATICS

MAT1C02 : Linear Algebra

Time : 3 Hours Max. Marks : 80

PART – A

Answer **four** questions from this Part. **Each** question carries 4 marks.

1. Let V be the real vector space of all functions f from $\mathbb{R}$ into $\mathbb{R}$. Check whether the set of all f such that $f(0) = f(1)$ is a subspace or not.

2. Prove that the only subspaces of $\mathbb{R}^1$ are $\mathbb{R}^1$ and the zero subspace.

3. Describe explicitly the linear transformation T from F^2 into F^2 such that $T(1, 0) = (a, b), T(0, 1) = (c, d)$.

4. Let T be a linear operator on $\mathbb{R}^3$ defined by $T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3)$. Prove or disprove that T is invertible.

5. In $\mathbb{R}^3$, let $\alpha_1 = (1, 0, 1)$, $\alpha_2 = (0, 1, -2)$, $\alpha_3 = (-1, -1, 0)$. If f is a linear functional on $\mathbb{R}^3$ such that $f(\alpha_1) = 1$, $f(\alpha_2) = -1$, $f(\alpha_3) = 3$ and if $\alpha = (a, b, c)$, find $f(\alpha)$.

6. Let V be an inner product space. The distance between two vectors α and β in V is defined by $d(\alpha, \beta) = ||\alpha - \beta||$. Then show that $d(\alpha, \beta) = d(\beta, \alpha)$.

PART – B

Answer **4** questions from this part without omitting **any** Unit. **Each** question carries **16** marks.

UNIT – I

7. a) Define basis of a vector space and give an example.

   b) Suppose P is an $n \times n$ invertible matrix over F. Let V be an n-dimensional vector space over F, and let $\mathcal{B}$ be an ordered basis of V. Then prove that there is a unique ordered basis $\mathcal{B}'$ of V such that

   $[\alpha]_{\mathcal{B}} = P[\alpha]_{\mathcal{B}'}$

   $[\alpha]_{\mathcal{B}'} = P^{-1}[\alpha]_{\mathcal{B}}$

   for every vector α in V.

8. a) Let V be an n-dimensional vector space over the field F and W be an n-dimensional vector space over F. Then prove that the space L(V, W) is finite-dimensional and has dimension mn.

   b) Let V and W be finite-dimensional vector spaces over the field F such that dim V = dim W. If T is a linear transformation from V into W, then prove that the following are equivalent.

   i) T is invertible

   ii) T is nonsingular

   iii) T is onto.

9. a) Let V be an n-dimensional vector space over the field F and W an m-dimensional vector space over F. For each pair of ordered bases $\mathcal{B}$, $\mathcal{B}'$ for V and W respectively, the function which assigns to a linear transformation T its matrix to $\mathcal{B}$, $\mathcal{B}'$ is an isomorphism between the space L(V, W) and the space of $m \times n$ matrices over the field F.

   b) Let V be a finite-dimensional vector space over the field F, and let $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ be a basis for V. Then prove that there is a unique dual basis $\mathcal{B}^* = \{f_1, \ldots, f_n\}$ for V^* such that $f_i(\alpha_j) = \delta_{i,j}$. Also prove that for each linear functional f on V, $f = \sum_{i=1}^{n} f(\alpha_i) f_i$ and for each vector α in V, $\alpha = \sum_{i=1}^{n} f_i(\alpha)\alpha_i$.

UNIT – II

10. a) Define characteristic value of a linear operator.

    b) Let T be a linear operator on a finite dimensional vector space V. If f is the characteristic polynomial for T, then prove that f(T) = 0.

11. a) Let V be a finite-dimensional vector space. What is the minimal polynomial for the identity operator on V ?

    b) Let T is any linear operator on a vector space V. If W is an invariant subspace for T, then show that W is invariant under every polynomial in T and for each α in V, the conductor S (α; W) is an ideal in the polynomial algebra F[x].

    c) If T is any linear operator on a vector space V, then show that the null space of T is invariant under T.

12. a) Let V be a finite-dimensional vector space over the field F and let T be a linear operator on V. Then prove that T is triangulable if and only if the minimal polynomial for T is a product of linear polynomials over F.

    b) Let $\mathcal{F}$ be a commuting family of diagonalizable linear operators on the finite-dimensional vector space V. Then prove that there exists an ordered basis for V such that every operator in $\mathcal{F}$ is represented in that basis by a diagonal matrix.

UNIT – III

13. a) Let T be a linear operator on the space V, and let $W_1, \ldots, W_k$ and $E_1, \ldots, E_k$ satisfies

    i) Each E_i is a projection

    ii) $E_iE_j = 0$ if $i \neq j$;

    iii) $I = E_1 + \ldots + E_k$;

    iv) the range of E_i is W_i.

    Then prove that a necessary and sufficient conditions that each subspace W_i be invariant under T is that T commutes with each of the projections E_i.