K21P 0560

Reg. No. : ....

Name : ....

First Semester M.Sc. Degree (CBSS – Reg./Suppl. (Including Mercy Chance)/Imp.) Examination, October 2020
(2017 Admission Onwards)
MATHEMATICS
MAT1C04 : Basic Topology

Time : 3 Hours Max. Marks : 80

Instructions : *1) Answer **any four** questions from Part – **A**. **Each** question carries 4 marks.*

*2) Answer **any four** questions from Part – **B**. Without omitting **any** Unit. **Each** question carries **16** marks.*

PART – A

1. Show that the lower limit topology on $\mathbb{R}$ is not the usual topology on $\mathbb{R}$.
2. Let X be a set and let d be the discrete metric on X. Show that (X, d) is complete.
3. Let A be a subset of the topological space (X, τ). Show that $\tau_A = \{U \cap A : U \in \tau\}$ is a topology on A.
4. Let (X_1, τ_1) and (X_2, τ_2) be second countable spaces and let τ be the product topology of $X = X_1 \times X_2$. Show that (X, τ) is second countable.
5. Let X be a set with more than one member and let τ be the discrete topology on X. Is (X, τ) connected ? Is it totally disconnected ? Why ?
6. Prove that the closed unit interval has the fixed point property.

PART – B

Unit – I

7. a) Define a basis for a topology. State and prove a necessary and sufficient condition for a subset of P(X) to be a basis for a topology on X.

   b) Let X = {1, 2, 3, 4, 5} and $\mathscr{S} = \{\{1\}, \{1, 2, 3\}, \{2, 3, 4\}, \{3, 5\}\}$. Is $\mathscr{S}$ a subbasis for a topology τ on X ? If so what is τ ?

   c) Prove that every metric space is first countable.

P.T.O.

8. a) Define a separable space and prove that every second countable space is separable.
   b) Give an example with justification of a separable space that is not second countable.
   c) Prove that every separable metric space is second countable.

9. a) State and prove Baire category theorem.
   b) Let (X, τ) be a first countable space. Let $\langle X_n \rangle$ be a sequence in X and $x \in X$. Prove that $\langle X_n \rangle$ clusters at x if and only if there is a subsequence of $\langle X_n \rangle$ that converges to x.
   c) Prove that metrizability is a topological property.

Unit – II

10. a) If τ is the usual topology on $\mathbb{R}$, find the subspace topology on the subset of all integers.
    b) Let (A, τ_A) be a subspace of a topological space (X, τ) and let B be a subset of A. Prove that the closure of B in (A, τ_A) is $A \cap \overline{B}$, where $\overline{B}$ is the closure of B in X.
    c) State and prove that pasting lemma.

11. a) Define the product space of two topological spaces (X_1, τ_1) and (X_2, τ_2) and show that $\mathscr{S} = \{\pi_1^{-1}(U) : U \in \tau_1\} \cup \{\pi_2^{-1}(V) : V \in \tau_2\}$ is a subbasis for the product topology on $X_1 \times X_2$.
    b) Let $X = \{1, 2, 3\}$, $Y = \{4, 5\}$, $\tau = \{\phi, \{1\}, \{1, 2\}, X\}$, $U = \{\phi, \{4\}, Y\}$. Find a subbasis $\mathscr{S}$ for the product topology on $X \times Y$. Also find the basis $\mathscr{B}$ that $\mathscr{S}$ generates.
    c) Let (X_1, d_1) and (X_2, d_2) be metric spaces, for each $i = 1, 2$, let τ_i be the topology on X_i generated by d_i. Prove that the product topology on $X = X_1 \times X_2$ is same as the topology on X generated by the product metric.

12. a) Let $\{(X_\alpha, \tau_\alpha) : \alpha \in \Lambda\}$ be a family of topological spaces and for each $\alpha \in \Lambda$, let $(A_\alpha, \tau_{A_\alpha})$ be a subspace of (X_α, τ_α). Prove that the product topology on $\prod_{\alpha \in \Lambda} A_\alpha$ is same as the subspace topology on $\prod_{\alpha \in \Lambda} A_\alpha$ determined by the product topology on $\prod_{\alpha \in \Lambda} X_\alpha$.

   b) Let $\{(X_\alpha, \tau_\alpha) : \alpha \in \Lambda\}$ be an indexed family of first countable spaces and let $X = \prod_{\alpha \in \Lambda} X_\alpha$. Prove that the product space (X, τ) is first countable if and only if τ_α is the trivial topology for all but a countable number of α.

Unit – III

13. a) Prove that a topological space (X, τ) is connected if and only if it cannot be expressed as the union of two non-empty sets that are separated in X.
    b) Let τ be the usual topology on $\mathbb{R}$. Show that $(\mathbb{R}, \tau)$ is connected.
    c) Prove that fixed point property is a topological invariant.

14. a) Let $\{(X_\alpha, \tau_\alpha) : \alpha \in \Lambda\}$ be a collection of topological spaces and suppose that for each $\alpha \in \Lambda$. $X_\alpha \neq \phi$. Let $X = \prod_{\alpha \in \Lambda} X_\alpha$. Prove that the product space (X, τ) is connected if and only if for each $\alpha \in \Lambda$, (X_α, τ_α) is connected.
    b) Define a pathwise connected space and show that the topologist's sine curve is not pathwise connected.

15. a) Define a locally pathwise connected space. Prove that a topological space is locally pathwise connected if and only if each path component of each open set is open.
    b) Define (i) totally disconnected space, (ii) 0-dimensional space (iii) T_0 – space.
    c) Prove that every 0-dimensional space is totally disconnected.
