K21P 0559

Reg. No. : ....

Name : ....

First Semester M.Sc. Degree (CBSS – Reg./Suppl. (Including Mercy Chance)/Imp.) Examination, October 2020
(2017 Admission Onwards)
MATHEMATICS
MAT1C03 : Real Analysis

Time : 3 Hours Max. Marks : 80

Instructions : *Answer **any four** questions from Part **A**. **Each** question carries **4** marks. Answer **any four** questions from Part **B**, without omitting any **Unit**. **Each** question carries **16** marks.*

PART – A

1. Let X be an infinite set and define $d : X \times x \to \mathbb{R}$ by $d(x, x) = 0$ for all $x \in X$ and $d(x, y) = 1$ if $x, y \in X$ and $x \neq y$. Prove that d is a metric on X.
2. Let f be a continuous real function on a metric space X. Let Z (f) be the set of all $p \in X$ at which $f(p) = 0$. Prove that Z(f) is closed.
3. Define a monotonically increasing function. If $f'(x) > 0$ in (a, b), prove that f is strictly increasing in (a, b).
4. Suppose $f \geq 0$, f is continuous on (a, b) and that $\int_a^b f(x)dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.
5. If $f \in R(\alpha)$ on [a, b], prove that $|f| \in R(\alpha)$ and $\left|\int_a^b f\,d\alpha\right| \leq \int_a^b |f|\,d\alpha$.
6. Let f and g be complex valued function defined by $f(t) = e^{2\pi it}$ if $t \in [0, 1]$, $g(t) = e^{2\pi it}$ if $t \in [0, 2]$. Prove that f and g have the same graph but are not equivalent.

PART – B

Unit – I

7. a) Let $\{E_n\}$, n = 1, 2, 3, ..., be a sequence of countable sets. Prove that $\bigcup_{n=1}^{\infty} E_n$ is countable.
   b) Prove that the set of all sequences whose elements are the digits 0 and 1 is uncountable.
   c) Let X be a metric space and $K \subset Y \subset X$. Prove that K is compact relative to X if and only if K is compact relative to Y.

P.T.O.

8. a) Prove that every k-cell is compact.

   b) Define a connected set in a metric space X. Prove that a subset E of $\mathbb{R}^1$ is connected if and only if it has the following property :
   if $x \in E$, $y \in E$ and $x < z < y$, then $z \in E$.

9. a) Let f be a continuous mapping of a compact metric space X into a metric space Y. Prove that f is uniformly continuous on X.

   b) Define discontinuity of the second kind. Illustrate with an example.

   c) Prove that a monotonic function has no discontinuities of the second kind.

Unit – II

10. a) State and prove the generalized mean value theorem.

    b) Let f be a real differentiable function on [a, b] and that $f'(a) < \lambda \leq f'(b)$. Prove that there is a point $x \in (a, b)$ such that $f'(x) = \lambda$.

    c) Let $\bar{f}$ be a continuous mapping of [a, b] into $\mathbb{R}^k$ and $\bar{f}$ be differentiable in (a, b). Prove that there exists $x \in (a, b)$ such that $|\bar{f}(b)-\bar{f}(a)| \leq (b-a)|\bar{f}'(x)|$.

11. a) State and prove Taylor's theorem.

    b) Prove that $f \in R(\alpha)$ on [a, b] if and only if for every $\epsilon > 0$ there exists a partition P such that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$.

12. a) If f is monotonic on [a, b] and if α is continuous on [a, b], prove that $f \in R(\alpha)$.

    b) If $f_1, f_2 \in R(\alpha)$ on [a, b], prove that $f_1 + f_2 \in R(\alpha)$ and that $\int_a^b (f_1+f_2)\,d\alpha = \int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha$.

    c) Define unit step function I. If $a < s < b$, f is bounded on [a, b], f is continuous at s and $\alpha(x) = I(x - s)$, prove that $\int_a^b f\,d\alpha = f(s)$.

Unit – III

13. a) State and prove the fundamental theorem of calculus.

    b) Define the Riemann-stieltjes integral of a mapping $\bar{f} = (f_1, f_2, ..., f_k)$ of [a, b] into $\mathbb{R}^k$. If $\bar{f} \in R(\alpha)$ for some monotonically increasing function α on [a, b], prove that $|\bar{f}| \in R(\alpha)$ and $\left|\int_a^b \bar{f}\,d\alpha\right| \leq \int_a^b |\bar{f}|\,d\alpha$.

    c) If f is of bounded variation on [a, b], prove that f is bounded on [a, b].

14. a) Let f be continuous on [a, b]. If f′ exists and is bounded on (a, b), prove that f is of bounded variation on [a, b].

    b) Determine whether f given by $f(x) = x^2 \sin(1/x)$ if $x \neq 0$, $f(0) = 0$ is of bounded variation on [0, 1].

    c) Let f be of bounded variation on [a, b]. Let $c \in (a, b)$. Prove that f is of bounded variation on [a, b] and $V_f(a, b) = V_f(a, c) + V_f(c, b)$.

15. a) Let f be of bounded variation on [a, b]. Let V be defined by $V(x) = V_f(a, x)$ for $a < x \leq b$ and $V(a) = 0$. Prove that V and V – f are increasing functions on [a, b].

    b) Let f and V be as in part (a). Prove that every point of continuity of f is also a point of continuity of V. Also prove that the converse is true.

______