K20P 0346

Reg. No. : ....

Name : ....

II Semester M.Sc. Degree (CBSS – Reg./Suppl./Imp.) Examination, April 2020
(2017 Admission Onwards)
MATHEMATICS
MAT 2C06 : Advanced Abstract Algebra

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any** 4 questions. **Each** question carries 4 marks.

1. Find all the units in $\mathbb{Z}[i]$.
2. Prove that $\sqrt{1+\sqrt{3}}$ is algebraic of degree 4 over $\mathbb{Q}$.
3. State Euclidean algorithm.
4. Show algebraically that it is possible to construct an angle of 30°.
5. Find the splitting field of $x^3 - 2$ over $\mathbb{Q}$.
6. Describe the group of the polynomial $(x^3 - 1) \in \mathbb{Q}[x]$ over $\mathbb{Q}$. (4×4=16)

PART – B

Answer 4 questions without omitting **any** Unit. **Each** question carries 16 marks.

Unit – I

7. a) State and prove Gauss's Lemma. 5
   b) Prove that if D is a UFD, then D[x] is a UFD. 7
   c) Prove that every Euclidean domain is a PID. 4
8. a) Prove that $\mathbb{Z}[i]$ is an Euclidean domain. 8
   b) Let p be an odd prime in $\mathbb{Z}$. Prove that $p = a^2 + b^2$ for integers a and b in $\mathbb{Z}$ if and only if $p \equiv 1 \pmod 4$. 8
9. a) Let F be a field and let f(x) be a non constant polynomial in F[x]. Prove that there exists an extension field E of F and an $\alpha \in E$ such that $f(\alpha) = 0$. 13
   b) Show that $x^3 + x^2 + 1$ is irreducible over $\mathbb{Z}_2$. 3

P.T.O.

Unit – II

10. a) If E is a finite extension of a field F and K is a finite extension of E, then prove that K is a finite extension of F and $[K : F] = [K : E]\,[E:F]$. 11

b) Prove that a field is algebraically closed if and only if every non constant polynomial in $F[x]$ factors in $F[x]$ into linear factors. 5

11. a) Prove that the set of all constructible real numbers forms a subfield F of the field of real numbers. 10

b) Prove that the field $GF(p^n)$ of p^n elements exists for every prime power p^n. 6

12. a) Prove that complex zeros of polynomials with real coefficients occur in conjugate pairs. 5

b) Let F be a finite field of characteristic p. Prove that the map $\sigma_p : F \to F$ defined by $\sigma_p(a) = a^p$ for $a \in F$ is an automorphism of F. Also prove that $F_{\{\sigma_p\}} \simeq \mathbb{Z}_p$. 7

c) Find primitive 10^{th} roots of unity in $\mathbb{Z}_{11}$. 4

Unit – III

13. a) Prove that a field E, where $F \leq E \leq \bar{F}$ is a splitting field over F if and only if every automorphism of $\bar{F}$ leaving F fixed maps E onto itself and thus induces an automorphism of E leaving F fixed. 12

b) If $F \leq E \leq K$, where K is a finite extension field of the field F, then prove that $\{K:F\} = \{K:E\}\{E:F\}$. 4

14. a) Prove that every field of characteristic zero is perfect. 5

b) State and prove primitive element theorem. 8

c) Find the degree of the splitting field of $x^4 - 1$ in $\mathbb{Q}[x]$ over $\mathbb{Q}$. 3

15. a) State main theorem of Galois theory. 6

b) Let K be a finite normal extension of F and let E be an extension of F, where $F \leq E \leq K \leq \bar{F}$. Prove that K is a finite normal extension of E and $G(K/E)$ is precisely the subgroup of $G(K/F)$ consisting of all those automorphisms that leave E fixed. Also prove that two automorphisms σ and τ in $G(K/F)$ induce the same automorphism of E onto a subfield of $\bar{F}$ if and only if they are in the same left coset of $G(K/E)$ in $G(K/F)$. 10

$(4\times16=64)$