K20P 0349

Reg. No. : ....

Name : ....

II Semester M.Sc. Degree (CBSS – Reg./Suppl./Imp.) Examination, April 2020
(2017 Admission Onwards)
MATHEMATICS
MAT2C09 : Foundations of Complex Analysis

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any four** questions. **Each** question carries **4** marks :

1. If $\gamma : [0, 1] \to \mathbb{C}$ is a closed rectifiable curve and $a \notin \{\gamma\}$, prove that $\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$ is an integer.

2. State and prove Morera's theorem.

3. If f has an essential singularity at z = a, then prove that for every $\delta > 0$, $\{f[\text{ann}(a; 0, \delta)]\}^- = \mathbb{C}$.

4. Let f be analytic on an open set containing $\bar{B}(a, R)$ and is one-one in B(a, R).

   If $\Omega = f[B(a, R)]$ and γ is the circle $|z - a| = R$, prove that $f^{-1}(\omega)$ is defined for each ω in Ω.

5. If $\{f_n\}$ is a sequence in H(G) and f belongs to $C(G, \mathbb{C})$ such that $f_n \to f$ then prove that f is analytic and $f_n^{(k)} \to f^{(k)}$.

6. Suppose $|z| < 1$ and $p \geq 0$. Prove that, $|1 - E_p(z)| \leq |z|^{p+1}$. **(4×4=16)**

P.T.O.

PART – B

Answer **any four** questions from this part without omiting **any** Unit.
Each question carries **16** marks.

Unit – I

7. a) Let G be a connected open set and let $f : G \to \mathbb{C}$ be an analytic function. Prove the following are equivalent :
   i) $f \equiv 0$
   ii) there is a point a in G such that $f^{(n)}(a) = 0$ for each $n \geq 0$.
   iii) $\{z \in G : f(z) = 0\}$ has a limit point in G.

   b) Let γ be a closed rectifiable curve in $\mathbb{C}$. Prove that $n(\gamma; a)$ is constant for a belonging to a component of $\mathbb{C} - \{\gamma\}$.

8. a) Suppose f is analytic in $B(a, R)$ and let $f(a) = \alpha$. If $f(z) - \alpha$ has a zero of order m at $z = a$, prove that there exist $\epsilon > 0$ and $\delta > 0$ such that for $|\zeta - \alpha| < \delta$ the equation $f(z) = \zeta$ has exactly m simple roots in $B(a, \epsilon)$.

   b) State and prove Cauchy's Theorem-Third Version.

9. a) If G is simply connected and $f : G \to \mathbb{C}$ is analytic in G, prove that f has a primitive.

   b) State and prove Goursat's theorem.

Unit – II

10. a) State and prove Residue theorem.

    b) Show that $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$.

11. a) Let f be meromorphic in the region G with zeros $z_1, \ldots, z_n$ and poles $p_1, \ldots, p_m$ counted according to multiplicity. If g is analytic in G and γ is a closed curve in G with $\gamma \approx 0$ and not passing through any z_j or p_j, prove that

    $$\frac{1}{2\pi i}\int_\gamma g\frac{f'}{f} = \sum_{j=1}^{n} g(z_j)n(\gamma; z_j) - \sum_{j=1}^{m} g(p_j)n(\gamma; p_j)$$

    b) State and prove Rouche's theorem.

    c) State and prove maximum modulus theorem (First version).

12. a) State and prove Schwarz's Lemma.

    b) If $|a| < 1$ prove that the map ϕ_a defined by $\phi_a(z) = \frac{z-a}{1-\bar{a}z}$ is a bijective map from $D = \{z : |z| < 1\}$ to D. Also prove that ϕ_a maps ∂D to ∂D and $\phi_a'(a) = (1-|a|^2)^{-1}$.

Unit – III

13. a) If G is open in $\mathbb{C}$ then prove that there is a sequence $\{K_n\}$ of compact subsets of G such that $G = \bigcup_{n=1}^{\infty} K_n, K_n \subset \text{int} K_{n+1}$ and every compact subset of G is a subset of K_n for some n.

    b) Prove that for a given $\epsilon > 0$ there exists a $\delta > 0$ and a compact set $K \subset G$ such that for f and g in $C(G, \Omega)$ $\sup\{d(f(z), g(z)) : z \in K\} < \delta$ implies $\rho(f, g) < \epsilon$.

14. State and prove Arzela-Ascoli theorem.

15. a) Let G be a region which is not the whole plane and such that every non-vanishing analytic function on G has an analytic square root. If $a \in G$, prove that there exists a one-one analytic function f on G such that $f(a) = 0$ and $f(G) = D = \{z : |z| < 1\}$.

    b) Let $\text{Re } z_n > -1$. Prove that the series $\sum \log(1 + z_n)$ converges absolutely iff the series $\sum z_n$ converges absolutely. **(4×16=64)**