K19P 0356

Reg. No. : ....

Name : ....

II Semester M.Sc. Degree (Reg./Suppl./Imp.) Examination, April 2019
(2017 Admission Onwards)
MATHEMATICS
MAT 2C 06 : Advanced Abstract Algebra

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any 4** questions. **Each** question carries **4** marks.

1. Distinguish between primes and irreducibles of an integral domain.
2. Find all the units in $\mathbb{Z}\left[\sqrt{-5}\right]$.
3. Find $\left[\mathbb{Q}\left(\sqrt{2}, \sqrt[4]{2}\right) : \mathbb{Q}\right]$.
4. If α and β are constructible real numbers, prove that $\alpha\beta$ is also constructible.
5. Find two extensions E and K of $\mathbb{Q}$ such that $[E : \mathbb{Q}] > [K : \mathbb{Q}]$, but $|G(E/\mathbb{Q})| < |G(K/\mathbb{Q})|$.
6. Give the lattice diagram of intermediate fields of $\mathbb{Q}\left(\sqrt{2}, i\right)$ over $\mathbb{Q}$. **(4×4=16)**

PART – B

Answer **4** questions without omitting **any** Unit. **Each** question carries **16** marks.

Unit – I

7. a) Prove that an ideal <p> in a PID is maximal if and only if p is an irreducible. 8

   b) Prove or disprove, if F is a field and x, y are indeterminates, then

   i) F is a PID ii) F[x, y] is a PID. 8

8. a) Prove that every Euclidean domain is a PID. 6

   b) Prove that any two non zero elements of a PID have a gcd and that any gcd of a and b can be expressed in the form $\lambda a + \mu b$ for $\lambda, \mu \in D$. 7

   c) Find a gcd of $x^3 - x^2 - 2x + 2$ and $x^3 + x^2 - 2$ in $\mathbb{Q}[x]$. 3

P.T.O.

9. a) Let P be an odd prime in $\mathbb{Z}$. Prove that $p = a^2 + b^2$ for some integers a and b if and only if $p \equiv 1 \pmod 4$). **14**

b) How would you construct a field of 4 elements ? **2**

Unit – II

10. a) Prove that if E is a finite extension of F and K is a finite extension of E, then K is a finite extension of F. **10**

b) Let $F \leq E \leq K$, be fields such that E is a finite extension of F and K is an algebraic extension of E. Then prove or disprove :

i) K is an algebraic extension of F

ii) K is a finite extension of F. **6**

11. a) Prove that 'trisecting the angle is impossible'. **6**

b) Prove that if F is a finite field of characteristic p, then the polynomial $x^{p^n} - x$ has p^n distinct zeros in the algebraic closure of F. **6**

c) Find the number of primitive 8^{th} roots of unity in GF(9). **4**

12. a) Define Frobenius automorphism of a finite field. If F is a finite field of characteristic p, prove that the fixed field of its Frobenius automorphism is isomorphic to $\mathbb{Z}_p$. **6**

b) State and prove conjugation isomorphism theorem. **10**

Unit – III

13. Let E be an algebraic extension of a field F. Let σ be an isomorphism of F onto a field F′. If $\overline{F'}$ denotes an algebraic closure of F′, prove that σ can be extended to an isomorphism τ of E onto a subfield of $\overline{F'}$ such that $\tau(a) = \sigma(a)$ for all $a \in F$. **16**

14. a) Prove that every field of characteristic 0 is perfect. **5**

b) Prove that if E is a finite extension of F, then {E : F} divides [E : F]. **5**

c) Prove that the field $\mathbb{Q}(\sqrt[4]{2})$ is not a splitting field extension of $\mathbb{Q}$. **6**

15. a) Let F be a finite field and let E be a finite extension of F of degree n. Prove that K is a normal extension of F, the group $G(K/F) \simeq \mathbb{Z}_n$ and G(K/F) is generated by $\sigma_p{}^r$ where $\sigma_p{}^r(\alpha) = \alpha^{p^r}$ for $\alpha \in K$ and $p^r = |F|$. **(4+6)**

b) Obtain the one-to-one correspondence between the intermediate fields of the extension $\mathbb{Z}_2 \leq F$ and the subgroups of $G(F/\mathbb{Z}_2)$ as in the main theorem, if $F = \mathbb{Z}_2(\alpha)$, where α is a root of $x^4 + x + 1$ in $\mathbb{Z}_2$. **6**
