

K19P 0357

Reg. No. :
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Name : .....

# II Semester M.Sc. Degree (Reg./Suppl./Imp.) Examination, April 2019 (2017 Admission Onwards) MATHEMATICS

MAT 2C 07: Measure and Integration

Time: 3 Hours

Max. Marks: 80

## PART - A

Answer any four questions from this Part. Each question carries 4 marks.

- 1. Show that if  $m^*(A) = 0$ , then  $m^*(A \cup B) = m^*(B)$  for any B.
- Prove that the set of irrationals in the interval [1, 4] is Lebesgue measurable and has a measure 3.
- 3. Show that  $\int_{1}^{\infty} dx / x = \infty$ .
- 4. If f is non negative measurable function, then prove that f = 0 a.e. if and only if  $\int f dx = 0$ .
- Let [X,S,μ] be a measure space and E₁, E₂ ∈ S. Show that μ (E₁ Δ E₂) = 0 implies μ (E₁) = μ (E₂).
- 6. Show that if  $\mu$  (X)  $< \infty$  and  $0 , then <math>L^p(\mu) \subseteq L^q(\mu)$ . (4×4=16)

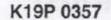
# PART - B

Answer any four questions from this Part without omitting any Unit. Each question carries 16 marks.

### Unit -

- 7. a) Prove that the outer measure of an interval is its length.
  - b) Prove that outer measure is translation invariant.
  - c) For any set A and any  $\epsilon > 0$ , show that there is an open set 0 containing A and such that  $m^*(0) \le m^*(A) + \epsilon$ .

P.T.O.





- 8. a) Prove that there exists a non measurable set.
  - b) Let T be a measurable set of positive measure and let  $T^* = [x y : x \in T, y \in T]$ . Show that T\* contains an interval  $(-\alpha, \alpha)$  for some  $\alpha > 0$ .
- 9. a) Let f be a non negative measurable function. Then prove that there exists a sequence  $\{\varphi_n\}$  of simple functions such that, for each x,  $\varphi_n(x) \uparrow f(x)$ .
  - b) Let f and g are non negative measurable functions. Then prove that  $\int f dx + \int g dx = \int (f + g) dx.$

# Unit - II

- 10. a) State and prove Lebesgue's dominated convergence theorem.
  - b) If f is Riemann integrable and bounded over the finite interval [a, b], then prove that f is integrable and  $R \int_{a}^{b} f dx = \int_{a}^{b} f dx$ .
- 11. a) Show that  $f \in L$  (a + h, b + h) and  $f_h(x) \equiv f(x + h)$ , then prove that  $f_h \in L(a, b)$  and  $\int_a^{b+h} f dx = \int_a^b f dx$ .
  - b) Let f be a bounded measurable function defined on the finite interval (a, b). Show that  $\lim_{\beta\to\infty}\int_a^b f(x)\sin\beta x\,dx=0$ .
  - c) Show that Lebesgue integrable function need not be Riemann integrable.
- 12. a) Let  $\mu^*$  be an outer measure on  $\mathcal{H}(\mathcal{R})$  and let  $S^*$  denote the class of  $\mu^*$  measurable sets. Then prove that S\* is a σ-ring and μ\* restricted to S\* is a complete measure.
  - b) Prove that  $\mu$  is  $\sigma$ -finite measure on a ring R, then prove it has a unique extension to the  $\sigma$ -ring  $S(\mathcal{R})$ .



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# Unit - III

- 13. a) Let  $[X, S, \mu]$  be a measure space and  $Y \in S$ . Let  $S_Y$  consist of those sets of S that are contained in S. Define  $\mu_{\nu}(E) = \mu(E)$  if  $E \in S_{\nu}$ . Then show that  $[Y, S_Y, \mu_Y]$  is a measure space.
  - b) Show that L<sup>p</sup>(μ) is a vector space.
- 14. a) State and prove Minkowski's inequality.
  - b) If  $\rho$  (f, g) =  $||f g||_p$  then prove that  $\rho$  is a metric on  $L^p(\mu)$ .
  - c) Let  $p \ge 1$  and let  $||f_n f||_p \to 0$ . Show that  $||f_n||_p \to ||f||_p$ .
- 15. a) Prove that if  $\{f_n\}$  is a sequence in  $L^{\infty}(\mu)$  such that  $\|f_n f_m\|_{\infty} \to 0$  as n,  $m \to \infty$ , then there exists a function f and such that  $\lim_{n \to \infty} f_n = f$  a.e.,  $f \in L^{\infty}(\mu)$  and  $\|f_n - f\|_{\infty} \to 0$ .
  - b) Let  $[X, S, \mu]$  be a measure space and  $E_n \in S$ , n = 1, 2, ... Show that
    - i)  $\mu$  (lim inf  $E_n$ )  $\leq$  lim inf  $\mu$ ( $E_n$ ).
  - ii) If  $\mu(X) < \infty$  then  $\limsup \mu(E_1) \le \mu$  ( $\limsup E_2$ ).

 $(4 \times 16 = 64)$