K18P 0134

Reg. No. : ....

Name : ....

Second Semester M.Sc. Degree (Supplementary/Improvement) Examination, March 2018
(2014-2016 Admn.)
MATHEMATICS
MAT 2C 06 : Advanced Abstract Algebra

Time : 3 Hours Max. Marks : 60

PART – A

Answer **any 4** questions. **Each** question carries 3 marks.

1. Prove that if f(x) is a primitive polynomial in $\mathbb{Z}[x]$ and if f(x) = g(x) h(x), for g(x), $h(x) \in \mathbb{Z}[x]$, then both g(x) and h(x) are also primitive.
2. Show that the element 3 is an irreducible in the integral domain $\mathbb{Z}\left[\sqrt{-5}\right] = \left\{a + i\sqrt{5}\, b : a, b \in \mathbb{Z}\right\}$
3. Give example of a real number which is not constructible. Justify your answer.
4. Let F be a field and $F \leq E$. If $\alpha, \beta \in E$ are algebraic over F, prove that $\alpha + \beta$ is algebraic over F.
5. Give examples for fields F and E such that $F \leq E$ and the degree [E : F] differs from the index {E : F}.
6. Describe the group of the polynomial $x^4 - 1$ over $\mathbb{Q}$. **(4×3=12)**

PART – B

Answer **any 4** questions without omitting **any** Unit. **Each** question carries **12** marks.

Unit – I

7. a) Define irreducible in an integral domain. Give an example of a non-constant polynomial in $\mathbb{Z}[x]$ which is an irreducible. **(1+2)**

   b) Prove that if D is a UFD, then D[x] is also a UFD. **9**

P.T.O.

8. a) Verify whether or not the function γ defined on $\mathbb{Z}$ by $\gamma(n) = n^2$ is a Euclidean norm for $\mathbb{Z}$. 3

b) State and prove the Fermat's $p = a^2 + b^2$ theorem. 9

9. a) Find an extension field F of $\mathbb{R}$ such that

i) F is an algebraic extension of $\mathbb{R}$.

ii) F is not an algebraic extension of $\mathbb{R}$.

Justify your answer in each. (2+2)

b) Let F be a field and $E = F(\alpha)$, where α is algebraic over F. If deg $(\alpha, F) = n \geq 1$, prove that every element β in $F(\alpha)$ can be uniquely expressed as $\beta = b_0 + b_1 \alpha + ... + b_{n-1} \alpha^{n-1}$, $b_i \in F$. 8

Unit – II

10. a) Prove that every finite extension of a field is an algebraic extension. 4

b) Find a basis for $\mathbb{Q}\left(\sqrt{2}, \sqrt{3}, \sqrt{6}\right)$ over $\mathbb{Q}$. 4

c) Prove that an algebraically closed field has no proper algebraic extension. 4

11. a) Prove that if α and $\beta \neq 0$ are constructible reals, then $\alpha\beta$ and α/β are also constructible. 8

b) Prove or disprove :

i) Every finite extension of a finite field is simple.

ii) Every finite extension of a finite field is algebraic. (2+2)

12. a) Let α be algebraic over F. Prove that every isomorphism, mapping $F(\alpha)$ onto a subfield of $\overline{F}$ and leaving F fixed, maps α onto a conjugate of α over F. 3

b) Consider the field $\mathbb{Z}_2[x]/\langle x^2 + x + 1\rangle \simeq \mathbb{Z}_2(\alpha) = \{0, 1, \alpha, 1+\alpha\}$; where α is a zero in $\overline{\mathbb{Z}}_2$ of $x^2 + x + 1$. Describe completely the Frobenius automorphism σ_2 of this field. What is the fixed field of σ_2 ? (4+2)

c) Is the map $\sigma_2 : \mathbb{Z}_3 \to \mathbb{Z}_3$ defined by $\sigma_2(\alpha) = \alpha^2$, an isomorphism ? Why ? 3

Unit – III

13. a) State isomorphism extension theorem. 2

b) Let E be a finite extension of F and σ, an isomorphism of F on to a field F'. Prove that the number of extensions of σ to an isomorphism of E onto a subfield of $\overline{F}'$ is finite and is independent of σ, F' and $\overline{F}'$. 8

c) Find all extension of the identity automorphism of $\mathbb{Q}$ to an isomorphism of $\mathbb{Q}\left(\sqrt[4]{2}\right)$ onto a subfield of $\overline{\mathbb{Q}}$. 2

14. a) Prove that a finite extension of F is separable over F if and only if every α in E is separable over F. 4

b) Prove that every finite field is separable. 8

15. a) Let K be a finite normal extension of a field F with Galois group G(K/F). Prove that for $F \leq E \leq K$, $[K : E] = |G(K/E)|$ and $[E : F] = (G(K/F) : G(K/E))$, where $(G(K/F) : G(K/E))$ is the index of the subgroup G(K/E) of G(K/F). Also prove that for $H \leq G(K/F)$, $G(K/K_H) = H$, where K_H is the fixed field of H in K. (4+5)

b) For an extension F of $\mathbb{Z}_3$ such that $[F : \mathbb{Z}_3] = 10$, what is the Galois group $G(F/\mathbb{Z}_3)$. 3