K18P 0229

Reg. No. : ....

Name : ....

Second Semester M.Sc. Degree (Regular) Examination, March 2018
MATHEMATICS
(2017 Admn.)
MAT 2C 07 : Measure and Integration

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any four** questions from this part. **Each** question carries **4** marks :

1. Prove that the outer measure of a countable set is 0.
2. For $k > 0$ and $A \subseteq \mathbb{R}$, let $kA = [kx : x \in A]$. Show that $m^*(kA) = km^*(A)$.
3. Let f be a measurable function and let $f = g$ a.e. Then prove that g is measurable.
4. Show that if μ is a σ-finite measure on R, then the extension $\bar{\mu}$ of μ to S^* is also σ-finite.
5. Show that if f, g $\in L^1(\mu)$, then prove that $|f^2 + g^2|^{\frac{1}{2}} \in L^1(\mu)$.
6. Show that $\lim \int_0^\infty \frac{dx}{(1+x/n)^n x^{1/n}} = 1$. **(4×4=16)**

PART – B

Answer **any four** questions from this part without omitting **any** unit. **Each** question carries **16** marks.

Unit – I

7. a) Let M be a class of Lebesgue measurable sets. Then prove that M is closed under the formation of countable unions.

   b) Prove that every interval is measurable.

P.T.O.

8. a) If $m^*(E) < \infty$ then prove that E is measurable if and only if, $\forall \varepsilon > 0, \exists$ disjoint finite intervals $I_1, I_2, \ldots, I_n$ such that $m^*(E \Delta \bigcup_{i=1}^{n} I_i) < \varepsilon$.
   b) Prove that Lebesgue measure is a regular measure.

9. a) Show that there exists a non measurable set.
   b) State and prove Fatou's Lemma.

Unit – II

10. a) If f is Riemann integrable and bounded over the finite interval [a, b], then prove that f is integrable and $R\int_a^b f\,dx = \int_a^b f\,dx$.
    b) Let f be bounded and measurable on a finite interval [a, b] and let $\varepsilon > 0$. Then prove that there exist
       i) A step function (a). If μ is measure on a ring R and if the set function $\mu^*(E)$ is define h such that $\int_a^b |f-h|\,dx < \varepsilon$.
       ii) A continuous function g such that g vanishes outside a finite interval and $\int_a^b |f-g|\,dx < \varepsilon$.

11. a) If μ is a measure on a ring R and if the set function $\mu^*(E)$ is defined on H (R) by $\mu^*(E) = \inf\left\{\sum_{n=1}^{\infty} \mu(E_n) : E_n \in R, n = 1, 2, \ldots, E \subseteq \bigcup_{n=1}^{\infty} E_n\right\}$. Then prove that μ^* is an outer measure on H(R).
    b) Let A, B be subsets of a set C, let A, B, C $\in$ R and let μ be a measure on R. Show that if $\mu(A) = \mu(C) < \infty$. Then prove that $\mu(A \cap B) = \mu(B)$.

12. a) If μ is σ-finite measure on R, then prove that it has a unique extension to the σ-ring S(R).
    b) Let S be the class of subsets of $\mathbb{R}$ such that E $\in$ S if either E or CE is at most countable. Show that S is a σ-ring.

Unit – III

13. a) Let $\{a_n\}$ be a sequence of non-negative numbers such that $a_n < \infty$, for each $n \in \mathbb{N}$ and for each $A \subseteq \mathbb{N}$, let $\mu(A) = \sum_{n \in A} a_n$. Show that $[\![\mathbb{N}, P(\mathbb{N}), \mu]\!]$ a σ-finite complete measure space.

    b) Let $[\![X, S, \mu]\!]$ be a measure space and $E_n \in S, n = 1, 2, \ldots$ show that
       i) $\mu(\liminf E_n) \le \liminf \mu(E_n)$
       ii) If $\mu(X) < \infty$ then $\limsup \mu(E_n) \le \mu(\limsup E_n)$.

14. a) Let $[\![X, S, \mu]\!]$ be a measure space and f a non negative measurable function. Then prove that $\phi(E) = \int_E f\,d\mu$ is a measure on the measurable space $[\![X, S]\!]$. Further prove that, if $\int f\,d\mu < \infty$, then $\forall \varepsilon > 0, \exists \delta > 0$ such that, if $A \in S$ and $\mu(A) < \delta$, then $\phi(A) < \varepsilon$.
    b) Let E and F be measurable sets, $f \in L(E)$ and $\mu(E \Delta F) = 0$. then prove that $f \in L(F)$ and $\int_E f\,d\mu = \int_F f\,d\mu$.

15. a) State and prove Holder's inequality.
    b) Prove that if $1 \le p < \infty$ and $\{f_n\}$ is a sequence in $L^p(\mu)$ such that $\|f_n - f_m\|_p \to 0$ as $n, m \to \infty$, then there exists a function f and a sequence $\{n_i\}$ such that $\lim f_{n_i} = f$ a.e. Further prove that $f \in L^p(\mu)$ and $\|f_n - f\|_p \to 0$. **(4×16=64)**
