K18P 0137

Reg. No. : ....

Name : ....

Second Semester M.Sc. Degree (Supplementary/Improvement) Examination, March 2018

(2014-2016 Admn.)

MATHEMATICS

MAT2C09 : Complex Analysis

Time : 3 Hours Max. Marks : 60

PART – A

Answer **any four** questions from this Part. **Each** question carries **3** marks. **(4×3=12)**

1. Let G be a region and define $G^* = \{z : \bar{z} \in G\}$. If $f : G \to \mathbb{C}$ is analytic, prove that $f^* : G^* \to \mathbb{C}$ defined by $f^*(z) = \overline{f(\bar{z})}$, is also analytic.

2. Evaluate $\int_r \frac{e^z - e^{-z}}{z^n} dz$ where n is a positive integer and $r(t) = e^{it}$, $0 \le t \le 2\pi$.

3. Let $f(z) = \frac{1}{(z-1)(z-2)}$. Obtain the Laurent expansion of f(z) in each of the annuli (i) ann (0; 1, 2) (ii) ann (0; 2, ∞).

4. Verify that $\int_0^\infty \frac{dx}{x^2+1} = \frac{\pi}{2}$.

5. Show that a necessary condition for the convergence of the infinite product with all its terms non-zero, is that the n^{th} terms tends to 1 as n tends to ∞.

6. Find the residue of the gamma function at each of its poles.

P.T.O.

PART – B

Answer **any four** questions from this part without omitting any unit. **Each** question carries 12 marks. **(4×12=48)**

UNIT – I

7. a) Let G be either the whole plane $\mathbb{C}$ or some open disk. If $u : G \to \mathbb{R}$ is a harmonic function, prove that u has a harmonic conjugate.

   b) Let $f : G \to \mathbb{C}$ be analytic and suppose that $\bar{B}(a;r) \subset G (r>0)$. If $r(t) = a + re^{it}$, $0 \le t \le 2\pi$, prove that $f(z) = \frac{1}{2\pi i}\int_{r}\frac{f(w)}{w-z}dw$ for $|z-a|<r$.

8. a) Let G be a connected open set and let $f : G \to \mathbb{C}$ be an analytic function. Prove that $f \equiv 0$ if and only if $\{z \in G : f(z) = 0\}$ has a limit point in G.

   b) State and prove Morera's theorem.

9. a) If r_0 and r_1 are two closed rectifiable curves in G and r_0 is homotopic to r_1, then prove that $\int_{r_0} f = \int_{r_1} f$ for every function f analytic on G.

   b) Let G be a region and suppose that f is a nonconstant analytic function on G. Prove that for any open set U in G, f(U) is open.

UNIT – II

10. a) State (without proof) the Laurent development. Use Laurent expansion to obtain characterizations for an isolated singularity to be (i) a removable singularity (ii) a pole of order m, (iii) an essential singularity.

    b) Show that $\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$.

11. a) State and prove the argument principle.

    b) Suppose f and g are meromorphic in a neighbourhood $\bar{B}(a, R)$ with no zeros and poles on the circle $r = \{z : |z-a| = R\}$. If Z_f, Z_g (P_f, P_g) are the number of zeros (poles) of f and g inside r counted according to their multiplicities and if $H(z) + g(z)|<|f(z)|+|g(z)|$ on r, prove that $Z_f - P_f = Z_g - P_g$. Also deduce the fundamental theorem of algebra.

12. a) State and prove the maximum modulus theorem (third version).

    b) State (without proof) Schwarz's lemma. If $D = \{z : |z| < 1\}$ and if $f : D \to D$ is a one to one analytic map of D onto itself with $f(a) = 0$, then prove that there is a complex number C with $|C| = 1$ such that $f = C\varphi_a$ where $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$, $|a|<1$.

UNIT – III

13. a) Define the set C (G, Ω). Show that it is never empty. Also show that it may contain only the constant functions.

    b) Prove that a set $F \subset C(G, \Omega)$ is normal if and only if its closure is compact.

    c) If $F \subset C(G, \Omega)$ is equicontinuous at each point of G, then prove that F is equicontinuous over each compact subset of G.

14. State and prove Riemann mapping theorem.

15. a) State (without proof) the Weierstrass factorization theorem. Derive the factorization $\sin \pi z = \pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$.

    b) Define the gamma function. Show that for $z \ne 0, -1, \ldots,$

    $$\Gamma(z) = \lim_{n\to\infty}\frac{n!\,n^z}{z(z+1)\ldots(z+n)} \text{ and } \Gamma(z+1) = z\Gamma(z).$$

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