K16P 0422

Reg. No. : ....

Name : ....

Second Semester M.Sc. Degree (Regular/Supplementary/Improvement) Examination, March 2016
(2014 Admn. Onwards)
MATHEMATICS
MAT 2C 07 – Measure and Integration

Time : 3 Hours   Max. Marks : 60

PART – A

Answer **four** questions from this Part. **Each** question carries **3** marks.

1. Show that the outer measure is translation invariant.

2. Show that the characteristic function χ_A of the set A is measurable if and only if A is measurable.

3. If f is an integrable function such that f = 0 a.e, then show that $\int f\,dx = 0$.

4. Show that the Lebesgue measure m defined on M, the class of measurable subsets of $\mathbb{R}$, is σ-finite and complete.

5. Let $\{f_n\}$ be a sequence of non-negative measurable functions, let $\lim f_n = f$ and $f_n \leq f$ for each n. Show that $\int f\,d\mu = \lim \int f_n\,d\mu$.

6. Let f, $g \in L^1(\mu)$; p, $q \in (0, 1)$ and p + q = 1. Show that $|f|^p\,|g|^q \in L^1(\mu)$.

P.T.O.

PART – B

Answer **any four** questions from this Part without omitting **any** Unit. **Each** question carries **12** marks.

UNIT – I

7. a) Prove that the class M of all Lebesgue measurable sets is a σ-algebra.

   b) Prove that every interval is measurable.

8. a) Show that there exists a non-measurable set.

   b) If f and g are real valued measurable functions on a measurable set E, prove that f + g and fg are measurable.

9. a) Let $\{f_n, n = 1, 2, ...\}$ be a sequence of non-negative measurable functions. Prove that $\liminf \int f_n \, dx \geq \int \liminf f_n \, dx$.

   b) If f and g are non-negative measurable functions, prove that $\int (f+g)\, dx = \int f \, dx + \int g \, dx$.

UNIT – II

10. a) State and prove the Lebesgue's dominated convergence theorem.

    b) State and prove a 'continuous parameter' version of dominated convergence theorem.

11. a) If f is Riemann integrable and bounded over the finite interval [a, b], prove that f is integrable and $R\int_a^b f(x)\,dx = \int_a^b f(x)\,dx$.

    b) Let f be a bounded measurable function defined on the finite interval (a, b). Show that $\lim_{\beta \to \infty} \int_a^b f(x) \sin \beta x \, dx = 0$.

12. a) Let $\{A_i\}$ be a sequence in a ring R. Prove that there is a sequence $\{B_i\}$ of disjoint sets of R such that $B_i \subseteq A_i$ for each i and $\bigcup_{i=1}^{N} A_i = \bigcup_{i=1}^{N} B_i$ for each N so that $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$.

    b) With usual notations prove that the outer measure μ^* on H(R) defined by μ on R and the corresponding outer measure $\bar{\mu}$ on S(R) and $\bar{\mu}$ on S^* are the same.

UNIT – III

13. a) Let $\{a_n\}$ be a sequence of non-negative numbers and for $A \subseteq \mathbb{N}$, let $\mu(A) = \sum_{n \in A} a_n$. Show that $[\mathbb{N}, P(\mathbb{N}), \mu]$ is a measure space.

    b) Let $[X, S, \mu]$ be a measure space and f a non-negative measurable function. Then prove that $\varphi(E) = \int_E f \, d\mu$ is a measure on the measurable space $[X, S]$. Further if $\int f \, d\mu < \infty$ then prove that for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $A \in S$ and $\mu(A) < \delta$, then $\varphi(A) < \varepsilon$.

14. a) Define the space $L^p(\mu)$. If $f, g \in L^p(\mu)$ prove that $af + bg \in L^p(\mu)$ where a and b are constants. Also if $\mu(X) < \infty$ and $0 < p < q < \infty$, then show that $L^q(\mu) \subseteq L^p(\mu)$.

    b) State and prove Minkowski's inequality.

15. For $1 \leq p \leq \infty$, prove that $L^p(\mu)$ is a complete metric space.

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