M 27243

Reg. No.: ....

Name: ....

II Semester M.A./M.Sc./M.Com. Degree (Reg./Sup./Imp.) Examination, March 2015
MATHEMATICS
(2014 Admn. Onwards)
MAT 2C 06 – Advanced Abstract Algebra

Time : 3 Hours Max. Marks : 60

PART – A

Answer **any 4** questions. **Each** question carries **3** marks.

1. Give an example of a UFD, that is not a PID. Justify your claim.
2. Prove that the integer 5 is not an irreducible as an element in $\mathbb{Z}[i]$.
3. Discuss the constructibility of the real numbers $\sqrt[3]{5}$ and $\sqrt[4]{5}$.
4. Find all the primitive 10^{th} roots of unity and 5^{th} roots of unity in $\mathbb{Z}_{11}$.
5. Prove that the field $\mathbb{Q}\left(\sqrt[3]{2}\right)$ is not a splitting field over $\mathbb{Q}$, where $\sqrt[3]{2}$ is the real cube root of 2.
6. Define (finite) separable extension. What are the separable extensions of $\mathbb{Q}$? Justify.

PART – B

Answer **4** questions without omitting **any** Unit. **Each** question carries **12** marks.

Unit – I

7. a) Prove that every Euclidean domain is a PID.

   b) Define prime and irreducible elements in an integral domain. Give an example to show that an integral domain can contain irreducibles that are not primes.

P.T.O.

8. a) Let D be a UFD and F, the field of quotients of D. Prove that a non constant f (x) $\in$ D [x] factors into a product of two polynomials of lower degrees r and s in F [x] if and only if it has a factorization into polynomials of same degrees r and s in D [x].

b) Show that the integral domain $\mathbb{Z}[\sqrt{-5}]$ is not a UFD. Give an example of a multiplicative norm on $\mathbb{Z}[\sqrt{-5}]$.

9. a) Let E be an algebraic extension of F and $\alpha \in E$. Prove that there exists an irreducible polynomial $p(x) \in F[x]$ such that $p(\alpha) = 0$.

b) Compute deg (1 + i, Q) and deg (π, Q(π^3)).

Unit – II

10. a) Prove that if E is a finite extension of a field F and K is a finite extension of E, then K is a finite extension of F and [K : F] = [K : E] [E : F].

b) Prove that the real fifth root of two, $\sqrt[5]{2}$ is not an element of $\mathbb{Q}(\sqrt[3]{2})$.

11. a) Prove that if F is any finite field, then for every positive integer n, there is an irreducible polynomial in F [x] of degree n.

b) Find an irreducible polynomial of degree 3 in $\mathbb{Z}_3$[x]. Describe an extension of $\mathbb{Z}_3$ containing a zero of this polynomial.

12. a) If F is a finite field of characteristic p, prove that the fixed field of the Frobenius automorphism of F is isomorphic to $\mathbb{Z}_p$.

b) Find all the conjugates of $\sqrt[4]{2}$ over Q. Also describe all the conjugation isomorphisms of $\mathbb{Q}(\sqrt[4]{2})$ onto subfields of $\overline{\mathbb{Q}}$. Which of them are automorphisms ?

Unit – III

13. a) Let E be a finite extension of a field F and σ be an isomorphism of F onto a field F′. Prove that the number of extensions of σ to an isomorphism of E onto a subfield of $\overline{F'}$ is finite and independent of F′, $\overline{F'}$ and σ.

b) For what extensions (finite) E of a field F, $\{E:F\} = |G(E/F)|$? Justify.

14. a) Prove that every finite field is perfect.

b) Find a primitive element for the extension $\mathbb{Q}(\sqrt{2}, \sqrt[3]{2})$ of Q.

15. a) Let K be a finite normal extension of F, with Galois group G (K/F). For a field E, where $F \leq E \leq K$, define λ (E) as the subgroup of G (K/F) leaving E fixed. Prove that λ is a one-one map of the set of all intermediate fields between F and K onto the set of all subgroups of G (K/F).

b) Give an example of a normal extension $F \leq K$ for which the Galois group G (K/F) is isomorphic to $\mathbb{Z}_{12}$.

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