K18P 1432

Reg. No. : ....

Name : ....

First Semester M.Sc. Degree (Reg./Suppl./Imp.) Examination, October 2018
(2017 Admn. Onwards)
MATHEMATICS
MAT1C04 : Basic Topology

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any four** questions. **Each** question carries **4** marks.

1. Let X be an infinite set and let $T = \{U \in P(X) : U = \varnothing \text{ or } X - U \text{ is finite}\}$. Prove that T is a topology on X.
2. Prove that the lower limit topology on $\mathbb{R}$ is strictly finer than the usual topology on $\mathbb{R}$.
3. Show by an example that separability is not a hereditary property.
4. For each $n \in \mathbb{N}$, let $X_n = \{1, 2\}$ and let T_n be the discrete topology on X_n. Let $X = \prod_{n \in \mathbb{N}} X_n$. Let T be the product topology on X and U be the box topology on X. Show that $T \neq U$.
5. Prove that the closed interval $I = [0, 1]$ has the fixed point property.
6. Show that the subspace of the real line (with usual topology) consisting of the rational numbers is a totally disconnected space. **(4×4=16)**

P.T.O.

PART – B

Answer **any four** questions without omitting **any** Unit. **Each** question carries **16** marks.

Unit – I

7. a) Give an example of a set X and topologies T_1 and T_2 on X such that $T_1 \cup T_2$ is not a topology on X.

   b) Let X be a set and let ζ be a collection of subsets of X such that $X = \cup\{S : S \in \zeta\}$. Prove that there is a unique topology T on X such that ζ is a subbasis for T.

   c) Let $X = \{1, 2, 3, 4, 5\}$, $\zeta = \{\{1\}, \{1, 2, 3\}, \{2, 3, 4\}, \{3, 5\}\}$. Find the topology generated by ζ.

8. a) With respect to the countable complement topology on a set X, prove that a proper subset of X is closed if and only if it is a countable set.

   b) Let A and B subsets of a topological space (X, T). Prove that

      i) $\bar{A} = A \cup A'$

      ii) A is closed if and only if $A = \bar{A}$.

      iii) $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

9. a) Prove that every complete metric space is of first category.

   b) Define a topological property and prove that metrizability is a topological property.

Unit – II

10. a) Let (A, T_A) be a subspace of a topological space (X, T). Prove that a subset C of A is closed in (A, T_A) if and only if there is a closed subset D of (X, T) such that $C = A \cap D$.

    b) Prove that every subspace of a separable metric space is separable.

    c) State the pasting lemma.

11. a) Let $\{(X_\alpha, T_\alpha) : \alpha \in \Lambda\}$ be an indexed family of topological spaces. Prove that the product topology is the weakest topology on $\prod_{\alpha \in \Lambda} X_\alpha$ for which each projection map $\pi_\beta : \prod_{\alpha \in \Lambda} X_\alpha \to X_\beta$ is continuous.

    b) Let $\{(X_\alpha, T_\alpha) : \alpha \in \Lambda\}$ be an indexed family of topological spaces and for each $\alpha \in \Lambda$, let $(A_\alpha, T_{A\alpha})$ be a subspace of (X_α, T_α). Prove that the product topology on $\prod_{\alpha \in \Lambda} A_\alpha$ is the same as the subspace topology on $\prod_{\alpha \in \Lambda} A_\alpha$ determined by the product topology on $\prod_{\alpha \in \Lambda} X_\alpha$.

12. a) Define weak topology. If U is the usual topology on $\mathbb{R}$, describe the weak topology T on $\mathbb{R}$ induced by the family that consists only of the function $i : \mathbb{R} \to (\mathbb{R}, U)$.

    b) Let (X_α, T_α) be an indexed family of topological spaces and let $X = \prod_{\alpha \in \Lambda} X_\alpha$. Prove that (X, T) is second countable if and only if (X_α, T_α) is second countable for all $\alpha \in \Lambda$ and T_α is the trivial topology for all but a countable number of α.

Unit – III

13. a) Prove that a topological space (X, T) is connected if and only if it can be expressed as the union of two non-empty sets that are separated in X.

    b) If T is the usual topology on $\mathbb{R}$, then prove that $(\mathbb{R}, T)$ is connected.

    c) Prove that the fixed point property is a topological invariant.

14. a) Prove that the topologist's sine curve is not pathwise connected.

    b) Define a locally pathwise connected space. If (X, T) is a connected, locally pathwise connected space, then prove that (X, T) is pathwise connected.

15. a) Define a locally connected space. Prove that a topological space (X, T) is locally connected if and only if each component of each open set is open.

    b) Let $\{(X_\alpha, T_\alpha) : \alpha \in \Lambda\}$ be a collection of topological spaces and T be the product topology on $X = \prod_{\alpha \in \Lambda} X_\alpha$. If for each $\alpha \in \Lambda$, (X_α, T_α) is locally connected and for all but a finite number of $\alpha \in \Lambda$, (X_α, T_α) is connected, then prove that (X, T) is locally connected. **(4×16=64)**