14. a) If there exists a Liapunov function E(x,y) for the system $\frac{dx}{dt} = F(x, y)$, $\frac{dy}{dt} = G(x, y)$, then prove that the critical point (0, 0) is stable. Further if $\frac{\partial E}{\partial x}F + \frac{\partial E}{\partial y}G$ is negative definite, then prove that the critical point (0, 0) is asymptotically stable.

b) Show that (0, 0) is a stable critical point of the system $\frac{dx}{dt} = -2yx$, $\frac{dy}{dt} = x^2 - y^3$.

15. a) Let f(x, y) be a continuous function that satisfies a Lipschitz condition $|f(x, y_1) - f(x, y_2)| \leq K|y_1 - y_2|$ on a strip $a \leq x \leq b$, $-\infty < y < \infty$. If (x_0, y_1) is any point of the strip prove that the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$ has a unique solution $y = y(x)$ on the interval $a \leq x \leq b$.

b) Show that $f(x, y) = xy^2$ satisfies a Lipschitz condition on any rectangle $a \leq x \leq b$, $c \leq y \leq d$.

K15P 0299

Reg. No. : ....

Name : ....

I Semester M.Sc. Degree (Reg./Sup./Imp.) Examination, November 2015
(2014 Admn. Onwards)
MATHEMATICS
MAT1C05 : Differential Equations

Time : 3 Hours Max. Marks : 60

PART – A

Answer **four** questions from this part. **Each** question carries **3** marks.

1. Find the solution of $y' = 2xy$ in the form of a power series $\sum_{n=0}^{\infty} a_n x^n$. Also solve the equation directly.

2. Find the indicial equation and its roots of $x^3y'' + (-1 + \cos 2x)y' + 2xy = 0$.

3. Using the generating relation $(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n$ where $P_n(x)$ is the n^{th} Legendre polynomial, show that $(n + 1) P_{(n+1)}(x) = (2n + 1) x P_n(x) - n P_{n-1}(x)$.

4. Find the general solution of the system $\frac{dx}{dt} = x + y$, $\frac{dy}{dt} = 4x - 2y$.

5. For the system $\frac{dx}{dt} = y(x^2 + 1)$, $\frac{dy}{dt} = 2xy^2$.

   i) Find the critical points

   ii) Find differential equation of the paths and

   iii) Solve this equation to find the paths.

6. State Picard's theorem. Explain how Lipschitz condition is related to Picard's theorem.

PART – B

Answer **any four** questions from this Part without omitting **any** Unit. **Each** question carries **12** marks.

Unit – I

7. a) Find the general solution of

$(1 + x^2)\, y'' + 2xy' - 2y = 0$ in terms of power series in x. Can you express this function in terms of elementary functions ?

b) For the equation $4xy'' + 2y' + y = 0$, verify that origin is a regular singular point and calculate two independent Frobenius series solutions.

8. a) Show that the equation $4x^2y'' - 8x^2y + 4(x^2 + 1)y = 0$ has only one Frobenius solution. Also find the general solution.

b) Find two linearly independent series solutions valid for $|x| < 1$ of the equation $(1 - x^2)\, y'' - xy' + p^2y = 0$, where p is a constant.

9. a) Define Gauss's hypergeometric equation and obtain the hypergeometric series as a solution of this equation.

b) Show that $\cos x = \lim_{a \to \infty} F\left(a, a, \frac{1}{2}, \frac{-x^2}{4a^2}\right)$.

Unit – II

10. a) Obtain the solution of the Legendre equation $(1 - x^2)y'' - 2xy' + n(n+1)y = 0$, where n is a nonnegative integer, bounded near x = 1 as a hypergeometric series.

b) Find the first three terms of the Legendre series of $f(x) = e^x$.

11. a) Prove that $\frac{d}{dx}\left[x^pJ_p(x)\right] = x^pJ_{p-1}(x)$ and $\frac{d}{dx}\left[x^{-p}J_p(x)\right] = -x^{-p}J_{p+1}(x)$. Also derive the recurrence formulas $2\,J_p'(x) = J_{p-1}(x) - J_{p+1}(x)$ and $\frac{2p}{x}J_p(x) = J_{p-1}(x) + J_{p+1}(x)$.

b) With usual notation, prove that

$$\int_0^1 x\, J_p(\lambda_m x) J_p(\lambda_n x)\, dx = \begin{cases} 0 & , \text{ if } m \neq n \\ \frac{1}{2} J_{p+1}(\lambda_n)^2, & \text{if } m = n \end{cases}.$$

12. a) If $x = x_1(t)$, $y = y_1(t)$ and $x = x_2(t)$, $y = y_2(t)$ are two linearly independent solutions on [a, b] of the system $\frac{dx}{dt} = a_1x + b_1y$, $\frac{dy}{dt} = a_2x + b_2y$, then prove that the general solution of the system is $y = c_1x_1(t) + c_2x_2(t)$, $y = c_1y_1(t) + c_2y_2(t)$, where c_1 and c_2 are any constants.

b) Solve the system $\frac{dx}{dt} = 3x - 4y$, $\frac{dy}{dt} = x - y$.

Unit – III

13. a) If the roots m_1 and m_2 of the auxiliary equation of the system $\frac{dx}{dt} = a_1x + b_1y$, $\frac{dy}{dt} = a_2x + b_2y$ where $a_1b_2 - a_2b_1 \neq 0$ are conjugate complex but not pure imaginary, then prove that the critical point (0,0) is a spiral.

b) Determine the nature and stability properties of the critical point (0, 0) of the system $\frac{dx}{dt} = -x - 2y$, $\frac{dy}{dt} = 4x - 5y$.