K15P 0298

Reg. No. : ....

Name : ....

I Semester M.Sc. Degree (Reg./Sup./Imp.) Examination, November 2015
(2014 Admn. Onwards)
MATHEMATICS
MAT 1C04 : Basic Topology

Time : 3 Hours Max. Marks : 60

PART – A

Answer **four** questions from this Part. **Each** question carries **3** marks.

1. Let X be an infinite set and let $T = \{U \in P(X) : U = \phi \text{ or } X - U \text{ is finite}\}$. Prove that T is a topology on X.

2. Is the usual topology on $\mathbb{R}$ second countable ? Justify your answer.

3. Define the subspace topology. If T is the usual topology on $\mathbb{R}$, describe the subspace topology on the set of all integers.

4. Let $X = \{1, 2, 3\}$, $T = \{\phi, \{1\}, \{1, 2\}, X\}$, $Y = \{4, 5\}$ and $U = \{\phi, \{4\}, Y\}$. Find a basis for the product topology on $X \times Y$.

5. Prove that the closed unit interval [0, 1] has the fixed point property.

6. Determine whether the real line with the usual topology is compact. **(4×3 =12)**

PART – B

Answer **four** questions from this Part without omitting any Unit. **Each** question carries **12** marks.

UNIT – I

7. a) Give an example of a set X and topologies T_1 and T_2 on X such that $T_1 \cup T_2$ is not a topology on X, justifying your claim.

   b) State and prove a necessary and sufficient condition for a family of subsets of a set X to be a basis for a topology on X.

P.T.O.

8. a) Let A be a subset of a topological space (X, T). Prove that A is closed if and only if $A' \subseteq A$.

   b) Let A and B be subsets of a topological space (X, T). Prove that

      i) $\overline{A} \subseteq \overline{B}$ whenever $A \subseteq B$.

      ii) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. Further show by an example that the inclusion in part (ii) cannot be replaced by an equality.

9. a) If d is the usual metric on $\mathbb{R}$, prove that ($\mathbb{R}$, d) is complete.

   b) Prove that metrizability is a topological property.

UNIT – II

10. a) Prove that the property of being a Hausdorff space is hereditary, but being separable is not hereditary.

    b) Prove that every subspace of a separable metric space is separable.

11. a) Let (X_1, T_1) and (X_2, T_2) be topological spaces and let $(X_1 \times X_2, T)$ be the product space. Prove that the projections $\pi_1 : X_1 \times X_2 \to X_1$ and $\pi_2 : X_1 \times X_2 \to X_2$ are continuous. Further prove that the product topology is the smallest topology for which both projections are continuous.

    b) Define box topology and product topology on the product of an indexed family of topological spaces. For each $n \in \mathbb{N}$ let $X_n = \{1, 2\}$ and let T_n be the discrete topology on X_n. Let $X = \prod_{n \in \mathbb{N}} X_n$ and T be the product topology on X and U be the box topology on X. Show that $T \neq U$.

12. a) Let (X, T) be a topological space, let $\{X_\alpha, T_\alpha) : \alpha \in \wedge\}$ be a collection of topological spaces and for each $\alpha \in \wedge$, let $f_\alpha : X \to X_\alpha$ be a continuous function. Prove that the collection $\{f_\alpha^{-1}(U_\alpha) : \alpha \in \wedge$ and $U_\alpha \in T_2\}$ is a basis for T if and only if $\{f_2 : \alpha \in \wedge\}$ separates points from closed sets.

    b) Let $\{(X_\alpha, T_\alpha) : \alpha \in \wedge\}$ be an indexed family of first countable spaces, and let $X = \prod_{\alpha \in \wedge} X_\alpha$. Prove that (X, T) is first countable if and only if T_α is the trivial topology for all but a countable number of α.

UNIT – III

13. a) Let T be the usual topology on $\mathbb{R}$. Prove that a subset of $\mathbb{R}$ is connected if and only if it is an interval.

    b) Prove that the topological sine curve is not pathwise connected.

14. a) Let (X, T) be a connected space, let O be an open cover of X, and let a, b be distinct points of X. Prove that there is a simple chain consisting of members of O that connects, a and b.

    b) Let (X, d) be a totally bounded complete metric space. Prove that (X, d) is compact.

15. a) Prove that every closed subset of a compact space is compact. Deduce that the Cantor set is compact.

    b) Prove that a topological space (X, T) is compact if and only if every family of closed subsets of X with the finite intersection property has a nonempty intersection. **(4×12 = 48)**