K15P 0297

Reg. No. : ....

Name : ....

I Semester M.Sc. Degree (Reg./Sup./Imp.) Examination, November 2015
(2014 Admission Onwards)
MATHEMATICS
MAT 1C03 : Real Analysis

Time : 3 Hours Max. Marks : 60

PART – A

Answer **four** questions from this Part. **Each** question carries **3** marks.

1. Give an example (with justification) of an open cover for the open interval (0, 1) with the usual metric, which has no finite subcover.

2. Suppose that f is a uniformly continuous mapping of a metric space X into a metric space Y. Prove that $\{f(x_n)\}$ is a Cauchy sequence in Y for every Cauchy sequence $\{x_n\}$ in X.

3. Suppose $f(x) = x \sin\frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. Show that $f'(x)$ exists for all $x \neq 0$ and that $f'(0)$ does not exist.

4. Suppose f is defined and differentiable for every x>0 and $f'(x) \to 0$ as $x \to +\infty$. Put $g(x) = f(x + 1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$.

5. If f(x) = 0 for all irrational x, f(x) = 1 for all rational x, prove that f is not Riemann integrable on [a, b] for any a < b.

6. Determine whether $f(x) = x^2 \cos\frac{1}{x}$ if $x \neq 0$ and $f(0) = 0$ is of bounded variation on [0, 1].

P.T.O.

PART – B

Answer **any four** questions from this Part without omitting any Unit. **Each** question carries **12** marks.

Unit – I

7. a) Prove that every infinite subset of a countable set is countable.

   b) Prove that set of all sequences whose elements are the digits 0 and 1 is uncountable.

   c) Show that the set of all rational numbers is countable.

8. a) Prove that every k-cell in $\mathbb{R}^k$ is compact.

   b) If E is an infinite subset of a compact set K, prove that E has a limit point in K.

   c) Prove that every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

9. a) Let f be a continuous mapping of a compact metric space X into e metric space Y. Prove that f is uniformly continuous on X.

   b) Define discontinuities of first and second kind of a function at a point and prove that a monotonic increasing function on (a, b) has no discontinuities of the first kind.

Unit – II

10. a) State and prove the generalized mean value theorem.

    b) If f is a continuous mapping of [a, b] into $\mathbb{R}^k$ and f is differentiable on [a, b]. Prove that there exists $x \in (a, b)$ such that $|f(b) - f(a)| \leq (b - a)|f'(x)|$.

11. a) Suppose (i) f is continuous for $x \geq 0$ (ii) $f'(x)$ exists for $x>0$ (iii) $f(0) = 0$ (iv) f' is monotonically increasing. Put $g(x) = \frac{f(x)}{x}$, $x>0$. Prove that g is monotonically increasing.

    b) If $f_1, f_2 \in R(\alpha)$ on [a, b], prove that $f_1 + f_2 \in R(\alpha)$ and that

$$\int_a^b (f_1 + f_2)d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha.$$

12. a) If f is continuous on [a, b], then prove that $f \in R(\alpha)$ on [a, b].

    b) Let f be a bounded real function on [a, b] and α be monotonically increasing such that $\alpha' \in R[a, b]$. Prove that $f \in R(\alpha)$ if and only if $f\alpha' \in R[a, b]$ and $\int_a^b f\, d\alpha = \int_a^b f(x)\alpha'(x)\, dx$.

Unit – III

13. a) Let $f \in R$ on [a, b]. Put $F(x) = \int_a^x f(t)dt$ for $a \leq x \leq b$. Prove that (i) F is continuous on [a, b] (ii) if f is continuous at $x_0 \in [a, b]$, prove that F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

    b) State and prove the fundamental theorem of calculus.

14. a) If f is continuous on [a, b] and if $|f'(x)| \leq A$ for all $x \in (a, b)$, then prove that f is of bounded variation on [a, b]. Is boundedness of f' a necessary condition for f to be of bounded variation ? Justify your answer.

    b) If f and g are of bounded variation on [a, b], prove that f+g and fg are of bounded variation on [a, b].

15. a) Let f be defined on [a, b]. Prove that f is of bounded variation on [a, b] if and only if f can be expressed as the difference of two increasing functions.

    b) Define a rectifiable path in $\mathbb{R}^n$ and length of a rectifiable path. If f is a rectifiable path defined on [a, b], prove that $\wedge_f(a, b) = \wedge_f(a, c) + \wedge_f(c, b)$ for $c \in (a, b)$.
