K15P 0296

Reg. No. : ....

Name : ....

I Semester M.Sc. Degree (Reg./Sup./Imp.) Examination, November 2015
(2014 Admn. Onwards)
MATHEMATICS
MAT 1C02 : Linear Algebra

Time : 3 Hours Max. Marks : 60

PART – A

Answer **any 4** questions. **Each** question carries **3** marks :

1. Find the nullspace of the linear operator T from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined by $T(x, y, z) = (2x, x + y, x + y + z)$. Is T invertible ? Why ?
2. Find a linear operator T on $\mathbb{R}^3$ such that $T^3 = 0$ but $T^2 \neq 0$.
3. Find all the characteristic values of the operator : $T(x_1, x_2) = (0, x_1 + x_2)$ defined on $\mathbb{C}^2$.
4. Let T be a linear operator on an n-dimensional vector space V. If $T^k = 0$ for some positive integer k, then prove that $T^n = 0$.
5. Prove that if E is a projection of a vector space V and $\beta \neq 0$ is a vector in the range of E, then the cyclic sub-space $Z(\beta; E)$ of V generated by β is one-dimensional.
6. Show that if (1) denotes an inner product defined on V and $(\alpha \mid \beta) = 0$ for all β in V, then $\alpha = 0$. **(4×3=12)**

PART – B

Answer **4** questions without omitting any Unit. **Each** question carries **12** marks :

Unit – I

7. a) Let T be a linear transformation from V into W. Prove that if T is invertible, then the inverse function T^{-1} is a linear transformation from W onto V. 5

P.T.O.

b) Let T be a linear operator on V with range R and null space N. Prove that the following are equivalent :

i) $R \cap N = \{0\}$

ii) If $T(T\alpha) = 0$,

then $T\alpha = 0$. **4**

c) Does there exist a linear transformation T from $\mathbb{R}^3$ into $\mathbb{R}^3$ whose range and null space are identical ? **3**

8. a) Let V be an n-dimensional vector space over F and $\mathcal{B} = \{\alpha_1, ..., \alpha_n\}$ be a basis for V. Prove that the linear functionals $f_1, ..., f_n$ defined by $f_i(\alpha_j) = \delta_{ij}$ form a basis for the dual space V* of V. Also show that, for each α in V,

$\alpha = \sum_{i=1}^{n} f_i(\alpha)\, \alpha_i$. **7**

b) Let W be the subspace of $\mathbb{R}^3$ spanned by (1, 0, 1) (0, 1, 1) and (2, –1, 1). Find a basis for W^0. **5**

9. a) Let g, $f_1, f_2, ..., f_r$ be linear operators on a vector space V with respective null spaces N, $N_1, ..., N_r$. Prove that g is a linear combination of $f_1, ..., f_r$ if and only if N contains the intersection $N_1 \cap ... \cap N_r$. **7**

b) Let f be the linear functional on F^2 defined by $f(x_1, x_2) = x_1 + x_2$. Describe the linear functional $T^t f$ if T is given by :

i) $T(x_1, x_2) = (x_1, 0)$

ii) $T(x_1, x_2) = (-x_1, x_2)$. **5**

Unit – II

10. a) Prove that if T is a linear operator on a finite-dimensional vector space V, then the characteristic and minimal polynomials for T have the same roots, except for multiplicities. **5**

b) If $T(x_1, x_2, x_3) = (3x_1 + x_2 - x_3, 2x_1 + 2x_2 - x_3, 2x_1 + 2x_2)$ is defined on $\mathbb{R}^3$, verify whether or not T is diagonalizable. **7**

11. State and prove the Cayley-Hamilton theorem. **12**

12. a) Let E be a projection on a finite-dimensional vectors pace V. Prove that :

i) A vector β is in the range of E if and only if $E\beta = \beta$

ii) $V = R \oplus N$, where R is the range and N is the nullspace of E, and

iii) E is diagonalizable. **8**

b) Find two proper subspaces W_1 and W_2 of $\mathbb{R}^2$ such that $\mathbb{R}^2 = W_1 \oplus W_2$. Also find a projection on $\mathbb{R}^2$ such that $R(E) = W_1$ and $N(E) = W_2$. **4**

Unit – III

13. a) Let T be a diagonalizable linear operator on a finite-dimensional space V, with the district characteristic values $C_1, ... C_k$. Prove that these exist linear operators $E_1, ..., E_k$ such that :

i) $T = C_1 E_1 + ... + C_k E_k$

ii) $I = E_1 + ... + E_k$

iii) $E_i E_j = 0$ for $i \neq j$

iv) $E_i^2 = E_i$ and

v) The range of E_i is the characteristic space for T associated with C_i. **9**

b) If $T(x_1, x_2, x_3) = (x_2, x_3, x_1)$, for $(x_1, x_2, x_3) \in \mathbb{R}^3$, prove that T has a cyclic vector. **3**

14. a) State cyclic decomposition theorem. **3**

b) Find all possible Jordan forms of a 5 × 5 complex matrix with characteristic polynomial $(x - 1)^3 (x + 2)^2$. **9**

15. a) Let V be an inner product space and $\beta_1, ... \beta_n$ be linearly independent vectors in V. Prove that there exist orthogonal vectors $\alpha_1, ..., \alpha_n$ in V such that for i = 1, 2, ..., x, the set $\{\alpha_1, ..., \alpha_i\}$ is a basis for the subspace spanned by $\{\beta_1, ..., \beta_i\}$. **6**

b) Let W be a subspace of the inner product space V and let $\beta \in V$. Prove that if $\alpha \in W$ is a best approximation to β by vectors in W, then $\beta - \alpha$ is orthogonal to every vector in W. **6**

(4×12=48)