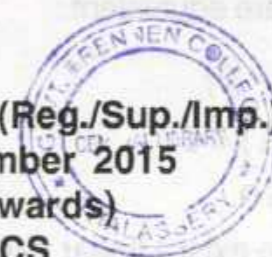




Reg. No. :

Name :



**I Semester M.Sc. Degree (Reg./Sup./Imp.)
Examination, November 2015
(2014 Admn. Onwards)
MATHEMATICS
MAT 1C02 : Linear Algebra**

Time : 3 Hours

Max. Marks : 60

PART - A

Answer **any 4** questions. **Each** question carries **3** marks :

1. Find the nullspace of the linear operator T from \mathbb{R}^3 to \mathbb{R}^3 defined by $T(x, y, z) = (2x, x + y, x + y + z)$. Is T invertible? Why?
2. Find a linear operator T on \mathbb{R}^3 such that $T^3 = 0$ but $T^2 \neq 0$.
3. Find all the characteristic values of the operator : $T(x_1, x_2) = (0, x_1 + x_2)$ defined on \mathbb{C}^2 .
4. Let T be a linear operator on an n -dimensional vector space V . If $T^k = 0$ for some positive integer k , then prove that $T^n = 0$.
5. Prove that if E is a projection of a vector space V and $\beta \neq 0$ is a vector in the range of E , then the cyclic sub-space $Z(\beta; E)$ of V generated by β is one-dimensional.
6. Show that if $(\cdot | \cdot)$ denotes an inner product defined on V and $(\alpha | \beta) = 0$ for all β in V , then $\alpha = 0$. (4x3=12)

PART - B

Answer **4** questions without omitting any Unit. **Each** question carries **12** marks :

Unit - I

7. a) Let T be a linear transformation from V into W . Prove that if T is invertible, then the inverse function T^{-1} is a linear transformation from W onto V . 5



- b) Let T be a linear operator on V with range R and null space N . Prove that the following are equivalent :
- $R \cap N = \{0\}$
 - If $T(T\alpha) = 0$, then $T\alpha = 0$.
- c) Does there exist a linear transformation T from \mathbb{R}^3 into \mathbb{R}^3 whose range and null space are identical ?
8. a) Let V be an n -dimensional vector space over F and $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$ be a basis for V . Prove that the linear functionals f_1, \dots, f_n defined by $f_i(\alpha_j) = \delta_{ij}$ form a basis for the dual space V^* of V . Also show that, for each α in V ,
- $$\alpha = \sum_{i=1}^n f_i(\alpha) \alpha_i.$$
- b) Let W be the subspace of \mathbb{R}^3 spanned by $(1, 0, 1)$, $(0, 1, 1)$ and $(2, -1, 1)$. Find a basis for W^0 .
9. a) Let g, f_1, f_2, \dots, f_r be linear operators on a vector space V with respective null spaces N, N_1, \dots, N_r . Prove that g is a linear combination of f_1, \dots, f_r if and only if N contains the intersection $N_1 \cap \dots \cap N_r$.
- b) Let f be the linear functional on F^2 defined by $f(x_1, x_2) = x_1 + x_2$. Describe the linear functional $T^1 f$ if T is given by :
- $T(x_1, x_2) = (x_1, 0)$
 - $T(x_1, x_2) = (-x_1, x_2)$.
- Unit - II**
10. a) Prove that if T is a linear operator on a finite-dimensional vector space V , then the characteristic and minimal polynomials for T have the same roots, except for multiplicities.
- b) If $T(x_1, x_2, x_3) = (3x_1 + x_2 - x_3, 2x_1 + 2x_2 - x_3, 2x_1 + 2x_2)$ is defined on \mathbb{R}^3 , verify whether or not T is diagonalizable.
11. State and prove the Cayley-Hamilton theorem.



12. a) Let E be a projection on a finite-dimensional vectors pace V . Prove that :
- A vector β is in the range of E if and only if $E\beta = \beta$
 - $V = R \oplus N$, where R is the range and N is the nullspace of E , and
 - E is diagonalizable.
- b) Find two proper subspaces W_1 and W_2 of \mathbb{R}^2 such that $\mathbb{R}^2 = W_1 \oplus W_2$. Also find a projection on \mathbb{R}^2 such that $R(E) = W_1$ and $N(E) = W_2$.

Unit - III

13. a) Let T be a diagonalizable linear operator on a finite-dimensional space V , with the distinct characteristic values C_1, \dots, C_k . Prove that there exist linear operators E_1, \dots, E_k such that :
- $T = C_1 E_1 + \dots + C_k E_k$
 - $I = E_1 + \dots + E_k$
 - $E_i E_j = 0$ for $i \neq j$
 - $E_i^2 = E_i$ and
 - The range of E_i is the characteristic space for T associated with C_i .
- b) If $T(x_1, x_2, x_3) = (x_2, x_3, x_1)$, for $(x_1, x_2, x_3) \in \mathbb{R}^3$, prove that T has a cyclic vector.
14. a) State cyclic decomposition theorem.
- b) Find all possible Jordan forms of a 5×5 complex matrix with characteristic polynomial $(x - 1)^3 (x + 2)^2$.
15. a) Let V be an inner product space and β_1, \dots, β_n be linearly independent vectors in V . Prove that there exist orthogonal vectors $\alpha_1, \dots, \alpha_n$ in V such that for $i = 1, 2, \dots, n$, the set $\{\alpha_1, \dots, \alpha_i\}$ is a basis for the subspace spanned by $\{\beta_1, \dots, \beta_i\}$.
- b) Let W be a subspace of the inner product space V and let $\beta \in V$. Prove that if $\alpha \in W$ is a best approximation to β by vectors in W , then $\beta - \alpha$ is orthogonal to every vector in W .

(4x12=48)