14. a) If there is a Liapunov function E (x, y) for the system $\frac{dx}{dt} = F(x, y)$, $\frac{dy}{dt} = G(x, y)$ then prove that the critical point is stable. Further show that if $\frac{\partial E}{\partial x}F + \frac{\partial E}{\partial y}G$ is negative definite, than (0, 0) is asymptotically stable.

b) Prove that the critical point (0, 0) of the linear system $\frac{dx}{dt} = a_1 x + b_1 y, \frac{dy}{dt} = a_2 x + b_2 y$ is a node if the roots of the auxiliary equation are equal.

15. a) State and prove Picards Theorem on the existence and uniqueness of solutions of the IVP : $y' = f(x, y), y(x_0) = y_0$.

b) Solve the following IVP by Picard's method

$$\begin{cases} \frac{dy}{dx} = z, & y(0) = 1 \\ \frac{dz}{dx} = -y, & z(0) = 1 \end{cases}$$

Reg. No.: ....

Name: ....

I Semester M.A./M.Sc./M.Com. Degree (Reg./Sup./Imp.) Examination, November 2014
MATHEMATICS (2013 & Earlier Admn.)
Paper – V : DIFFERENTIAL EQUATIONS

Time : 3 Hours Max. Marks : 60

Instruction : *Answer **any four** questions from Part **A**, and **four** questions from Part **B** without omitting **any** Unit.*

PART – A

Answer **any four** questions. **Each** question carries **3** marks.

1. Find the power series solution of $(1+x)y' = py$, $y(0) = 1$, where p is a constant.

2. Prove that the equation $x^2y'' - 3xy' + (4x + 4)y = 0$ has only one Frobenius series solution. Find it.

3. Prove that $\cos x = \lim_{a\to\infty} F(a, a, \frac{1}{2}, -x^2/4a^2)$.

4. Prove that $P_3(x) = \frac{1}{2}(5x^3 - 3x)$.

5. For the following non-linear system :

$$\begin{cases} \frac{dx}{dt} = e^x \\ \frac{dy}{dt} = e^x \cos x \end{cases}$$

1) Find the critical points
2) Find the differential equation of paths.
3) Solve this equation to find the paths.

6. Consider the initial value problem : $y' = |y|, y(x_0) = y_0$

i) For what points (x_0, y_0) does Picards theorem imply that this problem has a unique solution on some interval $|x - x_0| \le h$?

ii) For what points (x_0, y_0) does this problem actually have a unique solution on some interval $|x - x_0| \le h$?

PART – B

Answer **any four** questions without omitting **any** Unit. **Each** question carries **12** marks.

Unit – I

7. a) For the following equation, verify that the origin is a regular singular point and calculate two independent Frobenius series solutions : $2xy'' + (x+1)y' + 3y = 0$.

b) Determine the nature of the point $x = 0$ for the differentiate equation $x^4y'' + (\sin x)\, y = 0$.

8. a) Find the general solution of the Legendre's equation $(1-x^2)y'' - 2xy' + p(p+1)y = 0$

b) Prove that the indicial equation of the differential equation : $x^2y'' + xy' + x^2y = 0$ has only one root.

9. a) Find the general solution of the Gauss' hypergeometric equation $x(1-x)y'' + [e-(a+b+1)x]\, y' - aby = 0$, near the singular point $x = 0$.

b) Prove that $F'(a,b,c,x) = \frac{ab}{c}F(a+1, b+1, c+1, x)$.

Unit – II

10. a) Express $J_2(x)$, $J_3(x)$ and $J_4(x)$ in terms of $J_0(x)$ and $J_1(x)$.

b) Find the first three terms of the Legendre series of $f(x) = \begin{cases} 0 \text{ if } & -1 \le x < 0 \\ x \text{ if } & 0 \le x \le 1 \end{cases}$

11. a) Prove the Rodrigue's formula : $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$.

b) Prove that $\frac{2p}{x}J_p(x) = J_{p-1}(x) + J_{p+1}(x)$.

12. a) Consider the linear system $\frac{dx}{dt} = a_1(t)x + b_1(t)y$; $\frac{dy}{dt} = a_2(t)x + b_2(t)y$, where the coefficient functions are continuous on an interval I. If $x = x_1(t)$, $y = y_2(t)$ and $x = x_2(t), y = y_2(t)$ are two solutions of the system, there find an expression for the Wronskian w(t) of these solutions. Prove that these solutions are linearly independent solutions of the given system if and only if $w(t) \neq 0$ for all $t \in I$.

b) Find the general solution of the system

$$\begin{cases} \frac{dx}{dt} = x - 2y \\ \frac{dy}{dt} = 4x + 5y \end{cases}$$

Unit – III

13. a) Consider the equation of motion for the damped vibrations of a pendulum $\frac{d^2x}{dt^2} + \frac{l}{m}\frac{dx}{dt} + \frac{g}{a}\sin x = 0$. Investigate the stability properties of the critical point (0, 0) of the equivalent system.

b) Show that (0, 0) is an asymptotically stable critical point of $\begin{cases} \frac{dx}{dt} = -y - x^3 \\ \frac{dy}{dt} = x - y^3 \end{cases}$

but is an unstable critical point of $\begin{cases} \frac{dx}{dt} = -y + x^3 \\ \frac{dy}{dt} = x + y^3 \end{cases}$