M 26573

Reg. No. :

Name :

I Semester M.A./M.Sc./M.Com. Degree (Reg./Sup./Imp.) Examination,
November 2014

MATHEMATICS (2014 Admn. – under CBSS)

MAT1C02 – Linear Algebra

Time: 3 Hours

Max. Marks: 60

PART - A

Answer any 4 questions. Each question carries 3 marks.

- 1 Verify the rank-nullity theorem for the operator T from \mathbb{R}^3 to \mathbb{R}^3 defined by $T(x_1, x_2, x_3) = (0, x_1, x_2)$.
- 2. Describe the elements of the dual of the basis $\mathcal{P} = \{(1,0,-1),(1,1,0),(0,1,1)\}$ of \mathbb{C}^3 .
- 3. Find the minimal polynomial of the operator T $(x_1, x_2) = (x_2, x_1)$, defined on \mathbb{R}^3 .
- 4. Find all subspaces of \mathbb{R}^2 which are invariant under T, if T $(x_1, x_2) = (-x_2, x_1)$, for $(x_1, x_2) \in \mathbb{R}^2$.
- Let T be a diagonalizable linear operator on an n-dimensional vector space.Prove that if T has a cyclic vector, then T has n distinct characteristic values.
- 6. Consider \mathbb{R}^2 with the inner product $(\alpha \mid \beta) = x_1 y_1 x_2 y_1 x_1 y_2 + 4x_2 y_2$, where $\alpha = (x_1, x_2)$ and $\beta = (y_1, y_2)$. Find all vectors in \mathbb{R}^2 orthogonal to (1, 2).

PART-B

Answer 4 questions without omitting any Unit. Each question carries 12 marks.

UNIT-I

- a) Let V and W be vector spaces over the field F and T be a linear transformation from V into W. Prove that if V is finite dimensional, then rank (T) + nullity (T) = dim V.
 - b) Does these exist a linear transformation T from \mathbb{R}^3 into \mathbb{R}^3 such that the range and null space of T are identical? Why? Find such a linear operator on \mathbb{R}^2 .

P.T.O.

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- 8. a) Let V and W be vector spaces over F with dim V = n and dim W = m. Let

 be an ordered basis for V and

 '' be an ordered basis for W. Prove that for each linear transformation T: V→W, there exists an m × n matrix A over F such that [Tα]

 and T = A[α]

 for all α ∈ V.
- b) The rotation of the plane R² about the origin at an angle 90° is a linear operator on R². Find the matrix of this operator in the standard basis of R².
- a) If W₁ and W₂ are subspaces of a finite dimensional vector space, then prove that W₁= W₂ if and only if W₁⁰ = W₂⁰.
 - Define transpose of a linear transformation. Prove that a linear operator T and its transpose T^t have the same characteristic values.

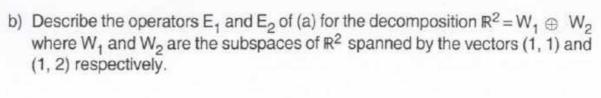
UNIT-II

- a) If T is a linear operator on a finite dimensional vector space V, with distinct characteristic values C₁, ..., C_k and if W_i is the null space of T – C_i I, prove that the following are equivalent.
 - i) T is diagonalizable
 - ii) The characteristic polynomial for T is $(x-C_1)^{d_1}...(x-C_k)^{d_k}$; where $d_i=\dim W_i,\,i=1,\,...,\,k$
 - iii) $Dim W_1 + ... + dim W_k = dim V$.

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- b) What is the characteristic polynomial of a nilpotent operator on an n-dimensional space? What can you say about its characteristic and minimal polynomials?
- 11. a) Prove that a linear operator on a finite-dimensional vector space is diagonalizable if and only if its minimal polynomial factors into a product of distinct linear polynomials over the field.
 - b) Let T be a linear operator on a vectorspace V. Prove that if every subspace of V is invariant under T, then T is a scalar multiple of the identity operator.
- 12. a) If $V = W_1 \oplus ... \oplus W_k$, prove that there exist k linear operators $E_1, ..., E_k$ on V such that
 - i) $E_{i}^{2} = E_{i}$
 - ii) $E_i E_j = 0$ for $i \neq j$
 - iii) $E_1 + ... + E_k = I$ and
 - iv) The range of E is W



UNIT - III

13. a) State primary decomposition theorem.

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b) Prove that if α is a non-zero vector in a finite dimensional vector space, then the degree of the T-annihilator of α is the dimension of the cyclic space Z (α ;T) generated by α . Also obtain a basis for Z(α ;T).

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- c) Let T be a linear operator on \mathbb{C}^2 . Prove that if T is not diagonalizable, then T is represented in some ordered basis by the matrix $\begin{bmatrix} C & 0 \\ 1 & C \end{bmatrix}$.
- State generalized Cayley-Hamilton theorem. Use the cyclic and primary decomposition theorems to prove it.

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- 15. a) If V is an inner product space, then prove that for any α, β in V and $C \in F$,
 - i) $|(\alpha|\beta)| \le |\alpha| |\beta|$ and
 - ii) $\|\alpha + \beta\| \le \|\alpha\| + \|\beta\|$.

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b) Prove that an orthogonal set of non-zero vectors is linearly independent.