M 26573

Reg. No. : ....

Name : ....

I Semester M.A./M.Sc./M.Com. Degree (Reg./Sup./Imp.) Examination, November 2014
MATHEMATICS (2014 Admn. – under CBSS)
MAT1C02 – Linear Algebra

Time : 3 Hours Max. Marks : 60

PART – A

Answer **any 4** questions. **Each** question carries **3** marks.

1. Verify the rank-nullity theorem for the operator T from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined by $T(x_1, x_2, x_3) = (0, x_1, x_2)$.

2. Describe the elements of the dual of the basis $\mathcal{B} = \{(1,0,-1), (1,1,0), (0,1,1)\}$ of C^3.

3. Find the minimal polynomial of the operator $T(x_1, x_2) = (x_2, x_1)$, defined on $\mathbb{R}^3$.

4. Find all subspaces of $\mathbb{R}^2$ which are invariant under T, if $T(x_1, x_2) = (-x_2, x_1)$, for $(x_1, x_2) \in \mathbb{R}^2$.

5. Let T be a diagonalizable linear operator on an n-dimensional vector space. Prove that if T has a cyclic vector, then T has n distinct characteristic values.

6. Consider $\mathbb{R}^2$ with the inner product $(\alpha \mid \beta) = x_1 y_1 - x_2 y_1 - x_1 y_2 + 4x_2 y_2$, where $\alpha = (x_1, x_2)$ and $\beta = (y_1, y_2)$. Find all vectors in $\mathbb{R}^2$ orthogonal to (1, 2).

PART – B

Answer **4** questions without omitting any Unit. **Each** question carries **12** marks.

UNIT – I

7. a) Let V and W be vector spaces over the field F and T be a linear transformation from V into W. Prove that if V is finite dimensional, then rank (T) + nullity (T) = dim V. 7

   b) Does these exist a linear transformation T from $\mathbb{R}^3$ into $\mathbb{R}^3$ such that the range and null space of T are identical ? Why ? Find such a linear operator on $\mathbb{R}^2$.

P.T.O.

8. a) Let V and W be vector spaces over F with dim V = n and dim W = m. Let $\mathscr{B}$ be an ordered basis for V and $\mathscr{B}'$ be an ordered basis for W. Prove that for each linear transformation $T : V \to W$, there exists an $m \times n$ matrix A over F such that $[T\alpha]_{\mathscr{B}'} = A[\alpha]_{\mathscr{B}}$, for all $\alpha \in V$.

b) The rotation of the plane $\mathbb{R}^2$ about the origin at an angle 90° is a linear operator on $\mathbb{R}^2$. Find the matrix of this operator in the standard basis of $\mathbb{R}^2$.

9. a) If W_1 and W_2 are subspaces of a finite dimensional vector space, then prove that $W_1 = W_2$ if and only if $W_1^0 = W_2^0$.

b) Define transpose of a linear transformation. Prove that a linear operator T and its transpose T^t have the same characteristic values.

UNIT – II

10. a) If T is a linear operator on a finite dimensional vector space V, with distinct characteristic values $C_1, \ldots, C_k$ and if W_i is the null space of $T - C_i I$, prove that the following are equivalent.

i) T is diagonalizable

ii) The characteristic polynomial for T is $(x - C_1)^{d_1} \ldots (x - C_k)^{d_k}$; where $d_i = \dim W_i$, $i = 1, \ldots, k$

iii) $\dim W_1 + \ldots + \dim W_k = \dim V$. **8**

b) What is the characteristic polynomial of a nilpotent operator on an n-dimensional space ? What can you say about its characteristic and minimal polynomials ? **4**

11. a) Prove that a linear operator on a finite-dimensional vector space is diagonalizable if and only if its minimal polynomial factors into a product of distinct linear polynomials over the field. **8**

b) Let T be a linear operator on a vectorspace V. Prove that if every subspace of V is invariant under T, then T is a scalar multiple of the identity operator. **4**

12. a) If $V = W_1 \oplus \ldots \oplus W_k$, prove that there exist k linear operators $E_1, \ldots, E_k$ on V such that

i) $E_i^2 = E_i$

ii) $E_i E_j = 0$ for $i \neq j$

iii) $E_1 + \ldots + E_k = I$ and

iv) The range of E_i is W_i **7**

b) Describe the operators E_1 and E_2 of (a) for the decomposition $\mathbb{R}^2 = W_1 \oplus W_2$ where W_1 and W_2 are the subspaces of $\mathbb{R}^2$ spanned by the vectors (1, 1) and (1, 2) respectively. **5**

UNIT – III

13. a) State primary decomposition theorem. **3**

b) Prove that if α is a non-zero vector in a finite dimensional vector space, then the degree of the T-annihilator of α is the dimension of the cyclic space $Z(\alpha; T)$ generated by α. Also obtain a basis for $Z(\alpha; T)$. **6**

c) Let T be a linear operator on $\mathbb{C}^2$. Prove that if T is not diagonalizable, then T is represented in some ordered basis by the matrix $\begin{bmatrix} C & 0 \\ 1 & C \end{bmatrix}$.

14. State generalized Cayley-Hamilton theorem. Use the cyclic and primary decomposition theorems to prove it. **12**

15. a) If V is an inner product space, then prove that for any α, β in V and $C \in F$,

i) $|(\alpha|\beta)| \leq \|\alpha\| \|\beta\|$ and

ii) $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$. **7**

b) Prove that an orthogonal set of non-zero vectors is linearly independent. **5**
