28. If f is a nonnegative function in M (X, **X**), then prove that there exists a sequence (φ_n) in M(X, **X**) such that

   i) $0 \leq \varphi_n(x) \leq \varphi_{n+1}(x)$ for $x \in X$, $n \in \mathbb{N}$.

   ii) $f(x) = \lim \varphi_n(x)$ for each $x \in X$.

   iii) Each φ_n has only a finite number of real values.

29. Let μ be a measure defined on a σ-algebra **X**.

   i) If (E_n) is an increasing sequence in **X**, then prove that $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim \mu(E_n)$.

   ii) If (F_n) is a decreasing sequence in **X** and if $\mu(F_1) < \infty$, then prove that $\mu\left(\bigcap_{n=1}^{\infty} F_n\right) = \lim \mu(F_n)$.

30. Let (f_n) be a sequence of integrable functions which converges almost everywhere to a realvalued measurable function f. If there exists an integrable function g such that $|f_n| \leq g$ for all n, then prove that f is integrable and $\int f\, d\mu = \lim \int f_n\, d\mu$. **(2×6=12)**

Reg. No. : ....

Name : ....

V Semester B.Sc. Hon's (Mathematics) Degree (Reg./Supple./Improv.) Examination, November 2020

(2016 Admission Onwards)

BHM 505(A) : INTEGRAL EQUATIONS AND MEASURE THEORY

Time : 3 Hours Max. Marks : 60

SECTION – A

Answer **any 4** questions out of 5 questions. **Each** question carries **1** mark.

1. Define Volterra integral equation of second kind.
2. Define symmetric Kernel.
3. When do you say a Kernel is separable ?
4. Define a measure.
5. Define a measure space and give an example. **(4×1=4)**

SECTION – B

Answer **any 6** questions out of 9 questions. **Each** question carries **2** marks.

6. Show that the function $\phi(x) = 1$ is a solution of the Fredholm integral equation $\phi(x) + \int_0^1 x e^{(x\xi - 1)} \phi(\xi) d\xi = e^x - x$.
7. Solve the homogeneous Fredholm integral equation $\phi(x) = \lambda \int_0^1 e^x e^\xi \phi(\xi) d\xi$.
8. Find the eigenvalues and eigenfunctions of $y(x) = \lambda \int_0^{2\pi} \sin x \cos \xi \, y(\xi) d\xi$.
9. Define the limit superior of a sequence of real numbers and illustrate with an example.

10. Define a σ-algebra and give an example.

11. Explain the concept of almost everywhere with a suitable example.

12. Define the integral of a simple function.

13. Define $M^+(X, \mathbf{X})$.

14. State Lebesgue Dominated Convergence Theorem. **(6x2=12)**

SECTION – C

Answer **any 8** questions out of 12 questions. **Each** question carries **4** marks :

15. Form the Volterra integral equation corresponding to the initial value problem $y'' + \lambda y(x) = f(x);\ y(0) = 1,\ y'(0) = 0$.

16. Find the Green's function of the boundary value problem $\frac{d^2y}{dx^2} + \lambda y = 0$ with $y(0) = 0$ and $y(1) = 0$.

17. Write down the four properties that have to be satisfied by Green's function of a second order differential equation with homogeneous boundary conditions.

18. Solve the integral equation $y(x) = 1 + \lambda \int_0^1 (-3x\xi) y(\xi) d\xi$, by using the iterative method.

19. Define the positive part and negative part of a function and hence find the positive and negative parts of $f(x) = \cos x$ defined over the interval $[0, \pi]$.

20. Prove that the following statements are equivalent for a function f on X to $\mathbb{R}$:
    i) For every $\alpha \in \mathbb{R}$, the set $A_\alpha = \{x \in X : f(x) > \alpha\}$ belongs to **X**.
    ii) For every $\alpha \in \mathbb{R}$, the set $B_\alpha = \{x \in X : f(x) \leq \alpha\}$ belongs to **X**.
    iii) For every $\alpha \in \mathbb{R}$, the set $C_\alpha = \{x \in X : f(x) \geq \alpha\}$ belongs to **X**.
    iv) For every $\alpha \in \mathbb{R}$ the set $D_\alpha = \{x \in X : f(x) < \alpha\}$ belongs to **X**.

21. Prove that an extended real-valued function f is measurable if and only if the sets $A = \{x \in X : f(x) = +\infty\}$, $B = \{x \in X : f(x) = -\infty\}$ belong to X and the real-valued function f_1 defined by $f_1(x) = \begin{cases} f(x), & \text{if } x \notin A \cup B, \\ 0, & \text{if } x \in A \cup B, \end{cases}$ is measurable.

22. If $\mu_1, \mu_2, \ldots, \mu_n$ are measures on **X** and $a_1, a_2, \ldots, a_n$ are nonnegative real numbers, then show that the function λ, defined for $E \in \mathbf{X}$ by $\lambda(E) = \sum_{j=1}^{n} a_j \mu_j(E)$, is a measure on **X**.

23. Let $g_n = n\chi_{\left[\frac{1}{n}, \frac{2}{n}\right]}$, $g = 0$. Show that $\int g\, d\lambda \neq \lim \int g_n\, d\lambda$. Does Fatou's Lemma apply ? Justify.

24. Let $f_n = \left(\frac{1}{n}\right) \chi_{[0,n)}$, $f = 0$. Show that the sequence (f_n) converges uniformly to f, but that $\int f\, d\lambda \neq \lim \int f_n\, d\lambda$. Why does this not contradict the Monotone Convergence Theorem ? Does Fatou's Lemma apply ?

25. Prove that a measurable function f belongs to L if and only if |f| belongs to L. Also prove that, in this case, $\left|\int f\, d\mu\right| \leq \int |f|\, d\mu$.

26. Prove that a constant multiple αf and a sum $f + g$ of functions in L belongs to L and $\int \alpha f\, d\mu = \alpha \int f\, d\mu$, $\int (f + g)\, d\mu = \int f\, d\mu + \int g\, d\mu$. **(8×4=32)**

SECTION – D

Answer **any 2** questions out of 4 questions. **Each** question carries **6** marks :

27. If the Kernel $K(x, \xi)$ is real and symmetric then prove that the eigenfunctions corresponding to the distinct eigenvalues of the homogeneous Fredholm integral equation $y(x) = \lambda \int_a^b K(x, \xi)\, y(\xi) d\xi$ are orthogonal over the interval (a, b).