Reg. No. : ....

Name : ....

VI Semester B.Sc. Degree (CCSS – Reg./Supple./Improv.) Examination, May 2016
CORE COURSE IN MATHEMATICS
6B11 MAT : Complex Analysis

Time : 3 Hours   Max. Weightage : 30
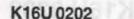
1. Fill in the blanks (weightage 1) :

    a) If $z_1 = 8 + 3i$ and $z_2 = 9 - 2i$, then $z_1/z_2 =$ _____

    b) If a function $f : \mathbf{C} \to \mathbf{C}$ is continuous at z_0, then $\lim\limits_{z \to z_0} f(z) =$ _____

    c) The singularities of $\dfrac{1}{\sin\left(\pi/z\right)}$ are _____

    d) If $f(z) = \sum\limits_{n=-\infty}^{\infty} a_n(z-a)^n$, then residue of $f(z)$ at $z = a$ is _____   **(W = 1)**

Answer **any six** questions from the following **nine** questions (weightage **one each**).

2. Reduce the quantity $\dfrac{1+2i}{3-4i} + \dfrac{2-i}{5i}$ to a real number.

3. Show that $|z - 1 + 3i| = 2$ represents a circle, find its centre and radius.

4. Show that $f(z) = \overline{z}$ is not differentiable, where $z = x + iy$.

5. Show that $u(x, y) = \dfrac{y}{x^2 + y^2}$ is harmonic.

6. Find the values of z such that $e^z = 1$.

7. Evaluate $\int_C \dfrac{z}{(9-z^2)(z+i)}\,dz$.

8. Prove that $\sin^{-1}(z) = -i\log\left[iz + (1-z^2)^{1/2}\right]$.

9. State the Cauchy's residue theorem.

10. Find the residue of $f(z) = \dfrac{z}{(z-1)(z+1)^2}$ at the poles. $\qquad$ $(6\times1=6)$

Answer **any 7** questions from the following **10** questions (weightage **2 each**).

11. Prove that an analytic function of constant absolute value is a constant.

12. Show that $u(x, y) = y^3 - 3x^2y$ is harmonic and find its harmonic conjugate.

13. If w(t), a complex valued function of a real variable, is integrable on [a, b], show

 that $\left|\int_a^b w(t)dt\right| \le \int_a^b |w(t)|dt$.

14. Find all the values of $(-8i)^{1/3}$.

15. Find $\int_C z^{1/2}dz$, where $z = 3e^{i\theta}$, $0 \le \theta \le \pi$.

16. Find the Laurent series of $f(z) = \dfrac{-1}{(z-1)(z-2)}$ in $1 < |z| < 2$.

17. If f(z) is analytic inside and on a positively oriented circle C with centre at z_0 and

 radius R, show that $\left|f^n(z_0)\right| \le \dfrac{n!\,M}{R^n}$ $(n = 1, 2, \dots)$, where M is a positive real number

 such that $|f(z)| \le M$.

18. If $z = z_0$ is a pole of order m of an analytic function f(z), show that
 $f(z) = (z - z_0)^m g(z)$, where g(z) is analytic and non-zero at z_0.

19. Show that $z = \dfrac{\pi i}{2}$ is a simple pole of $f(z) = \dfrac{\tanh z}{z^2}$ and find the residue of f(z) at

 this pole.

20. If two functions p and q are analytic at a point z_0, $p(z_0) \ne 0$, $q(z_0) = 0$ and

 $q'(z_0) \ne 0$, show that z_0 is a simple pole of the quotient $\dfrac{p(z)}{q(z)}$ and also prove that

 $\operatorname*{Res}_{z=z_0} \dfrac{p(z)}{q(z)} = \dfrac{p(z_0)}{q'(z_0)}$. $\qquad$ $(7\times2=14)$

Answer **any 3** questions from the following **5** questions (weightage **3 each**).

21. If $f(z) = u(x, y) + iv(x, y)$, $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0$, show that

 $\lim\limits_{z \to z_0} f(z) = w_0$ if and only if $\lim\limits_{(x, y) \to (x_0, y_0)} u(x, y) = u_0$ and

 $\lim\limits_{(x, y) \to (x_0, y_0)} v(x, y) = v_0$.

22. If $f(z) = u(x, y) + iv(x, y)$ is defined throughout some ε-neighbourhood of
 $z_0 = x_0 + iy_0$, u_x, u_y, v_x, v_y exist and are continuous everywhere in this
 neighbourhood and u and v satisfy the Cauchy-Riemann equations at (x_0, y_0),
 show that $f'(z_0)$ exists.

23. State and prove Cauchy's integral formula.

24. State and prove Liouville's theorem.

25. If f(z) is analytic throughout a disk $|z - z_0| < R_0$ centred at z_0 and with radius R_0,

 show that f(z) has the power series representation $f(z) = \sum\limits_{n=0}^{\infty} a_n(z-z_0)^n$, where

 $a_n = \dfrac{f^{(n)}(z_0)}{n!}$. $\qquad$ $(3\times3=9)$