M 9813

Reg. No. : ....

Name : ....

V Semester B.Sc. Degree (CCSS-Reg./Supple./Imp.) Examination, November 2015
CORE COURSE IN MATHEMATICS
5B07 MAT : Abstract Algebra

Time : 3 Hours　　　　　　　　　　　　　　　　Max. Weightage : 30

1. Mark each of the following **true** or **false**.
   a) If $*$ is any commutative binary operation on any set S, then $a * (b*c) = (b*c) * a$ for all $a, b, c \in S$.
   b) A binary operation on a set S assigns exactly one element of S to each ordered pair of elements of S.
   c) Every abelian group is cyclic.
   d) Every cyclic group has a unique generator. **(W=1)**

Answer **any six** questions from the following (weightage **one each**).

2. If $*$ is defined on Q^+, the set of all positive rationals, by $a*b = \frac{ab}{2}$, show that $(Q^+, *)$ is a group.

3. If G is a group with binary operation $*$, prove that $(a*b)' = b' * a'$, where a' is the inverse of a.

4. If A is any set and σ is a permutation of A, show that the relation '$\sim$' defined A by $a \sim b$ if and only if $b = \sigma^n(a)$, for some $n \in \mathbb{Z}$, $a, b \in A$, is an equivalence relation.

5. Write the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 6 & 7 & 4 & 1 & 5 & 2 \end{pmatrix}$ as a product of cycles.

6. Prove that every group of prime order is cyclic.

7. If H is a normal subgroup of a group G, show that the cosets of H in G forms a group under the binary operation (aH) (bH) = (ab) H.

8. Prove that a factor group of a cyclic-group is cyclic.

9. Define a ring homomorphism. Check whether $\varphi : \mathbb{Z} \to \mathbb{Z}$ defined by $\varphi(x) = 2(x)$ is a ring homomorphism.

10. What is the remainder when 8^{103} is divided by 13 ?

11. Define the characteristic of a ring. Obtain the characteristics of the rings $\mathbb{Z}_n$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$. **(W=6×1=6)**

Answer **any seven** questions from the following (weightage **two each**).

12. If G is a group binary operation $*$, show that $(a*b)' = a' * b'$ if and only if $a*b = b*a$, for a, b, $\in$ G, where a' is the inverse of a.

13. Prove that the intersection of two-subgroups H and K of a group G is a subgroup of G.

14. If G is a group and $a \in G$, show that $H = \{a^n / n \in \mathbb{Z}\}$ is the smallest subgroup of G that contains 'a'.

15. If H is a subgroup of a finite group G, prove that order of H is a divisor of order of G. Also prove that order of an element of a finite group divides the order of the group.

16. Obtain the group of symmetries of a square with vertices 1, 2, 3 and 4.

17. Define a homomorphism of a group G into a group G'. If $\varphi : G \to G'$ is a homomorphism of a group G on to a group G' and G is abelian, show that G' is also abelian.

18. If the map $\gamma : \mathbb{Z} \to \mathbb{Z}_n$ is difined by $\gamma(m) = r$, where r is the remainder given by the division algorithm when m is divided by n, show that γ is a homomorphism.

19. If $\varphi : G \to G'$ is a group homomorphism with Ker $\varphi = H$, prove that the set $\varphi^{-1}[\{\varphi(a)\}] = \{x \in G / \varphi(x) = \varphi(a)\}$ is the left coset aH of H.

20. Prove that the divisors of zero in $\mathbb{Z}_n$ are precisely those elements that are not relatively prime to n.

21. If R is a ring with units and if $n.1 \neq 0$ for all $n \in \mathbb{Z}^+$, prove that R has characteristic zero. If $n.1 = 0$, for some $n \in \mathbb{Z}^+$, prove that the characteristic of R is the smallest such n. **(W=7×2=14)**

Answer **any three** questions from the following (weightage **3 each**).

22. If G is a cyclic group with n elements and 'a' is a generator of G, prove that $b = a^s \in G$ generates a cyclic subgroup H of G containing n/d elements, where d is the g.c.d. of n and s.

23. Show that every group is isomorphic to a group of permutations.

24. Prove that the collection of all even permutations of $\{1, 2, \ldots, n\}$, $n \geq 2$, forms a subgroup of order $\frac{n!}{2}$ of the symmetric group S_n.

25. Prove that a subgroup H of a group G is a normal subgroup if and only if gH = Hg for all $g \in G$.

26. Show that the set G_n of non zero elements of $\mathbb{Z}_n$ that are not zero divisors forms group under multiplication modulo n. **(W=3×3=9)**