K18U 0323

Reg. No. : ....

Name : ....

VI Semester B.Sc. Hon's (Mathematics) Degree (Regular/Supple./Improv.) Examination, May 2018
BHM 603 : TOPOLOGY
(2013-15 Admissions)

Time : 3 Hours Max. Marks : 80

Answer **all** the 10 questions : **(10×1=10)**

1. Define the discrete metric on a non-empty set X.
2. Define the Hausdorff property of a metric space (X, d).
3. When we say that a topological space is metrizable ?
4. Define the co-finite topology.
5. If X = {a, b, c, d, e} with topology
   τ = {x, $\varnothing$, {a, b, c}, {c, d, e}, {c}}, check whether A = {a, d, e} is connected or not ?
6. Define the closure of a subset of a topological space.
7. 'Second countability is not preserved under continuous functions' - Give an example.
8. 'Length is not a topological property' - Give an example.
9. If X = {a, b} with topology τ = {X, $\varnothing$, {a}}, check whether (X, τ) is a T_1 – space.
10. If X = {a, b, c, d, e} and $\mathscr{A}$ = {{a, b, c,}, {c, d}, {d, e}}. Find the topology generated by $\mathscr{A}$.

P.T.O.

Answer **any 10** short answer questions out of **14** : **(10×3=30)**

11. Prove that $d(x, y) = |x - y|$ is a metric on $\mathbb{R}$. where $\mathbb{R}$ is the set of all real numbers.

12. Draw the figures of open unit ball B(0, 1) in the following three metric spaces.

    i) $(\mathbb{R}^2, d)$, where d is the usual metric on $\mathbb{R}^2$

    ii) $(\mathbb{R}^2, d_1)$, if $d_1(p, q) = \max\{|a_1 - b_1|, |a_2 - b_2|\}$, where $p = (a_1, a_2)$ and $q = (b_1, b_2)$ are arbitrary points in $\mathbb{R}^2$.

    iii) $(\mathbb{R}^2, d_2)$, if $d_2(p, q) = |a_1 - b_1| + |a_2 - b_2|$, where $p = (a_1, a_2)$, $q = (b_1, b_2)$.

13. Prove that any Cauchy sequence in a metric space is bounded.

14. Given any family $\mathscr{S}$ of subsets of X, prove that there is a unique topology τ on X-having $\mathscr{S}$ as a subbase.

15. Prove that a subset A of a topological space - X is dense in X if and only if for every non empty open subset B of X, $A \cap B \neq \emptyset$.

16. If A is a subset of a topological space (X, τ) prove that A is closed iff = $\bar{A} = A$.

17. If (X, τ) and (Y, u) are topological spaces and $f : X \to Y$ is a function, prove that f is continuous at $x_0 \in X$ iff for all $v \in u$, $f^{-1}(V) \in \tau$.

18. Prove that every quotient space of a discrete topological space is discrete.

19. Prove that every continuous image of a compact space is compact.

20. If $f : X \to Y$ is a continuous surjection, where X and Y are topological spaces and X is connected, prove that Y is also connected.

21. Define a T_0-space. Give an example.

22. Prove that regularity is a hereditary property.

23. If X is a Hausdorff space, $x \in X$ and F is a compact subset of X not containing x, show that there exist disjoint open sets U and V such that $x \in U$ and $F \subset V$.

24. Prove that every compact Hausdorff space is a T_3-space.

Answer **any 6** short essay questions out of **9** : **(6×5=30)**

25. If d is a metric on a non-empty set X, show that $\delta(x, y) = \frac{d(x,y)}{1+d(x,y)}$, where $x, y \in X$, is also a metric on X.

26. Let x, y be non-negative real numbers and p > 1 and q > 1 be defined in such a way that $\frac{1}{p} + \frac{1}{q} = 1$, prove that $xy \leq \frac{x^p}{p} + \frac{y^q}{q}$.

27. Prove that $\ell_p = \left\{ (a_n)_{n=1}^{\infty} : a_n \in \mathbb{R} \text{ or } \mathbb{C} \text{ and } \sum_{n=1}^{\infty} |a_n|^p < \infty \right\}$ is a vector space over $\mathbb{R}$ (or $\mathbb{C}$ as the caseman be).

28. Prove that a non empty open set in $\mathbb{R}$ is the union of a countable family of pairwise disjoint open intervals.

29. Prove that the limit of a sequence in a metric space is unique.

30. If X is a set and $\{\tau_i : i \in I\}$ is an indexed family of topologies on X, show that $\tau = \bigcap_{i \in I} \tau_i$ is a topology on X.

31. For a subset A of a topological space X, prove that $\bar{A} = A \cup A'$.

32. Prove that every quotient space of a locally connected space is locally connected.

33. Prove that all metric spaces are T_4.

Answer **any one** essay question out of **2** : **(1×10=10)**

34. If V is a an inner product space, prove that $|\langle x, y \rangle| \leq \|x\| \|y\|$ for all $x, y \in V$. Prove also that equality holds iff one of them is a scalar multiple of other.

35. Prove that every regular, Lindeloff space is normal.