K16U 1276

Reg. No. : ....

Name : ....

VI Semester B.Sc. Hon's (Mathematics) Degree (Regular) Examination, May 2016

BHM 601 : MEASURE AND INTEGRATION

Time : 3 Hours

Max. Marks : 80

Answer **all** the **10** questions : **(1×10=10)**

1. Define the characteristic function of a set E.
2. If φ is a simple function in $M^+(X, \mathbf{X})$, define $\int \varphi d\mu$.
3. Define the lower Riemann integral of a bounded function $f : [a, b] \to \mathbb{R}$.
4. Define measure μ.
5. Define a measurable space.
6. Define 'Borel Algebra'.
7. Define a simple function.
8. Define the indefinite integral of a function f with respect to a measure μ.
9. State Lebesgue Dominated Convergence theorem.
10. Define norm for a vector space V.

Answer **any 10**. Short answer questions out of **14**. **(10×3=30)**

11. Define the Lebesgue integral of a non-negative function f.
12. Define a σ-field and give an example.
13. Prove that any constant function is measurable.
14. If f is a measurable function, show that |f| is measurable.

P.T.O.

15. If λ is defined for E in $\mathbb{X}$ by $\lambda(E) = \int \varphi \chi_E d\mu$, show that λ is a measure on $\mathbb{X}$.

16. Given f belongs to $M^+(X, \mathbb{X})$, E and F belong to $\mathbb{X}$ and $E \subseteq F$, show that $\int_E f d\mu \leq \int_E d\mu$.

17. If f belongs to M^+ and $c \geq 0$, show that cf belongs to M^+ and $\int cf d\mu = c \int f d\mu$.

18. Suppose that f belongs to M^+ and λ on $\mathbb{X}$ is defined by $\lambda(E) = \int_E f d\mu$, show that λ is absolutely continuous with respect to μ.

19. If f belongs to L and $\lambda : \mathbb{X} \to \mathbb{R}$ is defined by $\lambda(E) = \int_E f d\mu$, prove that λ is a charge.

20. Prove that $\int (f + g)\, d\mu = \int f d\mu + \int g d\mu$.

21. If the function $t \to f(x, t)$ is continuous on [a, b] for each $x \in \mathbb{X}$ and $|f(x, t)| \leq g(x)$, where g(x) is an integrable function on $\mathbb{X}$, show that $F(t) = \int f(x, t)\, d\mu(x)$ is continuous for t in [a, b].

22. Prove that $N_u(f) = \int |f|\, d\mu$ is a semi-norm on $L(X, \mathbb{X}, \mu)$, where f belongs to the space $L(X, \mathbb{X}, \mu)$.

23. If the sequence (f_n) converges to f in Lp, show that it is a Cauchy sequence.

24. Describe L_∞ and define an essentially bounded function.

Answer **any 6** short essay questions out of **9** : **(6×5=30)**

25. Show that an extended real valued function is measurable if and only if the sets $A = \{x \in \mathbb{X} : f(x) = +\infty\}$ and $B = \{x \in \mathbb{X} : f(x) = -\infty\}$ belong to $\mathbb{X}$ and the real valued function f_1 defined by $f_1(x) = \begin{cases} f(x), & \text{if } x \notin A \cup B \\ 0, & \text{if } x \in A \cup B \end{cases}$, is measurable.

26. If μ is a measure defined on a σ–albegra $\mathbb{X}$, show that (i) $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim \mu(E_n)$, where $E_1 \subseteq E_2 \subseteq \ldots$ and (ii) $\mu\left(\bigcap_{n=1}^{\infty} Fn\right) = \lim \mu(F_n)$, where $F_1 \supseteq F_2 \supseteq \ldots$

27. Prove that f is measurable if and only if both f^+ and f^- are measurable.

28. If f belongs to M^+, show that f(x) = 0 if and only if $\int f d\mu = 0$.

29. If (f_n) is a monotone increasing sequence of functions in $M^+(X, \mathbb{X})$, which converges almost everywhere on $\mathbb{X}$ to a function f in M^+, show that $\int f d\mu = \lim \int f_n d\mu$.

30. Prove that a measurable function f belongs to L if and only if $|f| \in L$. Also, prove that $\left|\int f d\mu\right| \leq \int |f|\, d\mu$.

31. If the function $t \to f(x, t)$ is continuous on [a, b] for each $x \in X$ and $F(t) = \int f(x, t)\, d\mu(x)$, show that $\int_a^b F(t) dt = \int \left[\int_a^b f(x, t)\, dt\right] d\mu(x)$.

32. If $f \in L_p$, $g \in L_q$ where p >1, q >1 and $\frac{1}{p} + \frac{1}{q} = 1$, show that $fg \in L_1$ and $|fg|_1 \leq ||f||_p ||g||_q$.

33. Show that L_∞ is a complete normed linear space under the norm defined by $||f|| = \inf \{S(N), N \in \mathbb{X}, \mu(N) = 0\}$.

Answer **any one** essay question out of **2** : **(1×10=10)**

34. State and prove the monotone convergence theorem.

35. If $1 \leq p < \infty$, show that L_p is a complete normed linear space under the norm $||f||_p = \left(\int |f|^p d\mu\right)^{1/p}$.
