PART – D

(Essay Questions)

Answer **any 1** question **(34-35)**. **Each** question carries **10** marks. **(1×10=10)**

34. a) If ϕ and ψ are simple functions in $M^{+}(X, \chi)$ and $c \geq 0$, then show that

$$\int c\phi d\mu = c\int \phi d\mu, \text{ and } \int(\phi + \Psi)\, d\mu = \int \phi d\mu + \int \Psi d\mu$$

b) If λ is defined for E in χ by $\lambda(E) = \int \phi \chi_E\, d\mu$, then show that λ is a measure o χ.

35. a) State and prove Monotone Convergence Theorem.

b) If f, $g \in M^{+}$ and $c \geq 0$, then show that $f + g \in M^{+}$ and

$$\int(f+g)\, d\mu = \int f\, d\mu + \int g d\mu.$$

Reg. No. : ....

Name : ....

VI Semester B.Sc. Hon's (Mathematics) Degree (Supplementary) Examination, April 2019

(2013-'15 Admissions)

BHM 601 : MEASURE AND INTEGRATION

Time : 3 Hours Max. Marks : 80

PART – A

(Objective Type Questions)

Answer **all** questions **(1-10)**. **Each** question carries **1** mark. **(10×1=10)**

1. A finite linear combination of characteristic functions is called
2. Define Lebesgue integral of a nonnegative function f.
3. Define σ-algebra.
4. Define Borel σ-algebra.
5. Define positive and negative parts of a real function.
6. When we say that a measure is σ-finite ?
7. What do you mean by a measure concentrated at a point ?
8. Give an example of a measure which is σ-finite but not finite.
9. Define semi-norm.
10. Define Lebesgue space.

PART – B

(Short Answer Type)

Answer **any ten** questions **(11-24)**. **Each** question carries **3** marks. **(10×3=30)**

11. Define extended borel σ-algebra and show that it is a σ-algebra on $\mathbb{R}$.

12. Show that an extended real valued function f is measurable if and only if the sets $A = \{x \in X : f(x) = +\infty\}$, $B = \{x \in X : f(x) = -\infty\}$ belongs to the σ-algebra χ and the real valued function f_1 defined by $f_1(x) = \begin{cases} f(x) & \text{if } x \notin A \cup B \\ 0 & \text{if } x \in A \cup B \end{cases}$ is measurable.

13. Let $\{f_n(x)\}$ be a sequence in $M(X, \chi)$ and define the functions $f(x) = \inf f_n(x)$ and $F(x) = \sup f_n(x)$. Show that $f, F \in M(X, \chi)$.

14. Let (X, χ, μ) be a measure space and let $\{E_n\}$ be a sequence in χ. Then show that $\mu(\liminf E_n) \leq \liminf \mu(E_n)$.

15. If $f, g \in M^+(X, \chi)$ and $f \leq g$, then show that $\int f\, d\mu \leq g\, d\mu$.

16. If $f \in L$ and λ is defined on χ to $\mathbb{R}$ by $\lambda(E) = \int_E f\, d\mu$. Then show that λ is a charge.

17. Let $\{g_n\}$ be a sequence in M^+, then show that

$$\int\left(\sum_{n=1}^{\infty} g_n\right) d\mu = \sum_{n=1}^{\infty}\left(\int gn\, d\mu\right).$$

18. Show that a measurable function f belongs to L if and only if $|f| \in L$. Also show that $\left|\int f d\mu\right| \leq \int |f|\, d\mu$.

19. Let (X, χ, μ) be a measure space. Show that N_μ, defined by $N_\mu(f) = \int |f|\, d\mu$. $f \in L(X, \chi, \mu)$ is a semi-norm on the space $L(X, \chi, \mu)$.

20. Show that the Lebesgue space $L_1 = L_1(X, \chi, \mu)$ is a normed linear space.

21. If f and g are simple functions in $M(X, \chi)$, then show that $\phi = \sup\{f, g\}$ and $\psi = \inf\{f, g\}$ are also simple functions in $M(X, \chi)$.

22. State Cauchy-Bunyakovskii-Schwarz inequality and Minkowski's inequality.

23. Define Cauchy sequence and show that every convergent sequence in L_p space is Cauchy sequence.

24. Define L_∞ space. Also give a norm on L_∞ which is complete.

PART – C

(Short Essay Type)

Answer **any 6** questions **(25-33)**. **Each** question carries **5** marks. **(6×5=30)**

25. Show that the following statements are equivalent for a function $f : X \to \mathbb{R}$:

   a) For every $\alpha \in \mathbb{R}$, the set $A_\alpha = \{x \in X : f(x) > \alpha\} \in \chi$.

   b) For every $\alpha \in \mathbb{R}$, the set $B_\alpha = \{x \in X : f(x) \leq \alpha\} \in \chi$.

   c) For every $\alpha \in \mathbb{R}$, the set $C_\alpha = \{x \in X : f(x) \geq \alpha\} \in \chi$.

   d) For every $\alpha \in \mathbb{R}$, the set $D_\alpha = \{x \in X : f(x) < \alpha\} \in \chi$.

26. Let f and g are measurable real valued functions and let c be a real number. Then show that the functions cf, f^2, $f + g$, fg, $|f|$ are also measurable.

27. Let μ be a measure defined on a σ-algebra χ. If $\{E_n\}$ is an increasing sequence in χ, then show that $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim \mu(E_n)$

28. State and prove Fatou's lemma.

29. If μ is a charge on χ, let v be defined for $E \in \chi$ by $v(E) = \sup \sum_{j=1}^{n} |\mu(A_j)|$, where the supremum is taken over all finite disjoint collections $\{A_j\}$ in, χ with $E = \bigcup_j A_j$. Show that v is a measure on χ.

30. State and prove Lebesgue Dominated Convergence Theorem.

31. Suppose that f belongs to M^+. Then show that $f(x) = 0$ μ-almost everywhere on χ if and only if $\int f\, d\mu = 0$.

32. State and prove Holder's inequality in L_p space.

33. Suppose that for some $t_0 \in [a, b]$, the function $x \to f(x, t_0)$ is integrable on χ, that $\partial f/\partial t$ exists on $X \times [a, b]$, and that there exists an integrable function g on X such that $\left|\frac{\partial f}{\partial t}(x, t)\right| \leq g(x)$. Then show that the function F defined by $F(x) = \int f(x, t)\, d\mu(x)$ is differentiable on $[a, b]$.