K21U 0221

Reg. No. : ....

Name : ....

VI Semester B.Sc. Hon's (Mathematics) Degree (Supplementary) Examination, April 2021
(2014-2015 Admissions)
BHM 605(A) : DISCRETE FOURIER ANALYSIS

Time : 3 Hours Max. Marks : 80

Answer **all ten** questions : (10×1=10)

1. State Parseval's relation.
2. Define discrete Fourier transform.
3. When we say that a matrix is circulant ?
4. Describe first-stage wavelet basis for $\ell^2(\mathbb{Z}_N)$.
5. Define conjugate reflection of $w \in \ell^2(\mathbb{Z}_N)$.
6. Define p^{th}-stage wavelet basis for $\ell^2(\mathbb{Z}_N)$.
7. What do you mean by completeness property of $\ell^2(\mathbb{Z})$?
8. Define trigonometric system and trigonometric polynomial.
9. Define down sampling operator.
10. Define homogeneous wavelet system for $\ell^2(\mathbb{Z})$.

Answer **any 10** short answer questions out of 14 : (10×3=30)

11. Let $\hat{u} = \left(\sqrt{2}, 1, 0, 1\right)$ and $\hat{v} = \left(0, 1, \sqrt{2}, -1\right)$. Find $\{v, R_2u, R_2v\}$ for $\ell^2(Z_4)$.
12. State and prove Fourier inversion formula.
13. Let $w_N = e^{-2\pi i/N}$. Then prove that $\hat{z}(m) = \sum_{0}^{N-1} z(n)\omega_N^{mn}$.
14. For any $w \in \ell^2(\mathbb{Z}_N)$, prove that $w * \delta = w$. Where δ is the Dirac delta function.

P.T.O.

15. Suppose $z, w \in \ell^2(\mathbb{Z}_N)$. For any $k \in \mathbb{Z}$, then prove that $z * \tilde{w}(k) = \langle z, R_k w \rangle$.

16. Suppose $M \in \mathbb{N}$, $N = 2M$ and $z \in \ell^2(\mathbb{Z}_N)$. Define $z^* \in \ell^2(\mathbb{Z}_N)$ by $z^*(n) = (-1)^n z(n)$ for all n. Then prove that $(z^*)^\wedge(n) = \hat{z}(n + M)$ for all n.

17. Define Fourier series of $f \in L^1([-\pi, \pi))$.

18. Suppose N is divisible by 2^p. Suppose $u, v \in \ell^2(\mathbb{Z}_N)$ are such that the system matrix A(n) is unitary for all n. Let $u_1 = u$ and $v_1 = v$ and for all $\ell = 2, 3, \ldots, p$, define u_ℓ by equation $u_\ell(n) = \sum_{k=0}^{2^{\ell-1}-1} u_1\left(n + \frac{kN}{2^{\ell-1}}\right)$ and v_ℓ similarly with v_1 in place of u_1. Then prove that $u_1, v_1, u_2, v_2, \ldots, u_p, v_p$ is a p^{th}-stage wavelet filter sequence.

19. Prove that the trigonometric system is an orthonormal set in $L^2([-\pi, \pi))$.

20. Suppose H is a Hilbert space and $T : H \to H$ is a bounded linear transformation. Suppose the series $\Sigma_{n \in \mathbb{Z}} x_n$ converges in H. Then prove that $T\left(\sum_{n \in \mathbb{Z}} x_n\right) = \sum_{n \in \mathbb{Z}} T(x_n)$ where the series on the right converges in H.

21. For $z \in \ell^2(\mathbb{Z})$, prove that $z(n) = (\hat{z})^\vee(n)$.

22. Suppose $w, z \in \ell^1(\mathbb{Z})$. Prove that set $\{R_{2k}w\}_{k \in \mathbb{Z}}$ is orthonormal if and only if $|\hat{w}(\theta)|^2 + |\hat{w}(\theta + \pi)|^2 = 2$ for all $\theta \in [0, \pi)$.

23. Suppose $z, w \in \ell^2(\mathbb{Z})$. Prove that $U(z * w) = U(z) * U(w)$.

24. Suppose $z \in \ell^2(\mathbb{Z})$. Prove that $(U(z))^\wedge(\theta) = \hat{z}(2\theta)$ for all θ.

Answer **any 6** short answer questions out of 9 : **(6×5=30)**

25. Prove that the set $\{E_0, \ldots, E_{N-1}\}$ is an orthonormal basis for $\ell^2(\mathbb{Z}_N)$.

26. Let $z = (1, 1, 0, 2)$ and $w = (i, 0, 1, i)$ be vectors in $\ell^2(\mathbb{Z}_4)$. Find $z * w$.

27. Let $b \in \ell^2(\mathbb{Z}_N)$, and let T_b be the convolution operator associated with b. Then prove that T_b is translation invariant.

28. Suppose $M \in \mathbb{N}$, $N = 2M$ and $u \in \ell^2(\mathbb{Z}_N)$ is such that $\{R_{2k}u\}_{k=0}^{M-1}$ is an orthonormal set with M elements. Define $v \in \ell^2(\mathbb{Z}_N)$ by $v(k) = (-1)^{k-1}\overline{u(1-k)}$ for all k. Then prove that $\{R_{2k}v\}_{k=0}^{M-1} \cup \{R_{2k}u\}_{k=0}^{M-1}$ is a first-stage wavelet basis for $\ell^2(\mathbb{Z}_N)$.

29. Suppose N is even, say $N = 2M$, $z \in \ell^2(\mathbb{Z}_N)$, and $x, y, w \in \ell^2(\mathbb{Z}_{N/2})$. Then prove that $D(z) * w = D(z * U(w))$ and $U(x) * U(y) = U(x * y)$.

30. Suppose $f \in L^1([-\pi, \pi))$ and $\langle f, e^{in\theta} \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta = 0$ for all $n \in \mathbb{Z}$. Then prove that $f(\theta) = 0$ a.e.

31. Suppose $T : L^2([-\pi, \pi)) \to L^2([-\pi, \pi))$ is a bounded, translation-invariant linear transformation. Then prove that for each $m \in \mathbb{Z}$, there exists $\lambda_m \in \mathbb{C}$ such that $T(e^{im\theta}) = \lambda_m e^{im\theta}$.

32. Suppose $u \in \ell^1(\mathbb{Z})$ and $\{R_{2k}u\}_{k \in \mathbb{Z}}$ is orthonormal in $\ell^2(\mathbb{Z})$. Define a sequence $v \in \ell^1(\mathbb{Z})$ by $v(k) = (-1)^{k-1}\overline{u(1-k)}$. Then prove that $\{R_{2k}v\}_{k \in \mathbb{Z}} \cup \{R_{2k}u\}_{k \in \mathbb{Z}}$ is a complete orthonormal system (hence a first-stage wavelet system) in $\ell^2(\mathbb{Z})$.

33. Let $p \in \mathbb{N}$. For $\ell = 1, 2, \ldots, p$, suppose that $u_\ell, v_\ell \in \ell^1(\mathbb{Z})$ and the system matrix $A_\ell(\theta) = \frac{1}{\sqrt{2}}\begin{bmatrix} \hat{u}_\ell(\theta) & \hat{v}_\ell(\theta) \\ \hat{u}_\ell(\theta + \pi) & \hat{v}_\ell(\theta + \pi) \end{bmatrix}$ is unitary for all $\theta \in [0, \pi)$. Define $f_1 = v_1$, $g_1 = u_1$ and inductively, for $\ell = 2, 3, \ldots, p$ $f_\ell = g_{\ell-1} * U^{\ell-1}(v_\ell)$, $g_\ell = g_{\ell-1} * U^{\ell-1}(u_\ell)$. Define B as $B = \{R_{2^\ell} f_\ell : k \in \mathbb{Z}, \ell = 1, 2, \ldots, p\} \cup \{R_{2^p k} g_p : k \in \mathbb{Z}\}$.
Then prove that B is a complete orthonormal set (hence a p^{th}-stage wavelet system) for $\ell^2(\mathbb{Z})$.

Answer **any one** essay question out of 2 : **(1×10=10)**

34. Suppose $z \in \ell^2(\mathbb{Z}_N)$ and $k \in \mathbb{Z}$. Then prove that for any $m \in \mathbb{Z}$, $(R_k z)^\wedge(m) = e^{-2\pi imk/N}\hat{z}(m)$.

35. Suppose $M \in \mathbb{N}$, $N = 2M$, and $w \in \ell^2(\mathbb{Z}_N)$. Then prove that $\{R_{2k}w\}_{k=0}^{M-1}$ is an orthonormal set with M elements if and only if $|\hat{w}(n)|^2 + |\hat{w}(n + M)|^2 = 2$ for $n = 0, 1, \ldots, M - 1$.
