35. Suppose N is even, say N = 2M, $z \in l^2(\mathbb{Z}_N)$ and $x, y, w \in l^2(\mathbb{Z}_{N/2})$. Then prove that $D(z) * w = D(z * U(w))$ and $U(x) * U(y) = U(x * y)$.

36. Let $w \in l^2(\mathbb{Z}_N)$. Then prove that $\{R_k w\}_{k=0}^{N-1}$ is an orthonormal basis for $l^2(\mathbb{Z}_N)$ if and only if $|\hat{w}(n)| = 1 \; \forall n \in \mathbb{Z}_N$.

SECTION – D

(Answer **any 2** questions out of 6 questions. **Each** question carries **6** marks.) **(2×6=12)**

37. Let $T : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ be a translation-invariant linear transformation. Then prove that for each element of the Fourier basis F is an eigenvector of T. In particular, T is diagonalizable.

38. Prove that : Suppose $M \in \mathbb{N}$ and N = 2M. Let $u, v \in l^2(\mathbb{Z}_N)$. Then $B = \{R_{2k}v\}_{k=0}^{M-1} \cup \{R_{2k}u\}_{k=0}^{M-1}$ is an orthonormal basis for $l^2(\mathbb{Z}_N)$ if and only if the system matrix A(n) of u and v is unitary for each n = 0, 1, ..., M – 1. Equivalently, B is a first-stage wavelet basis for $l^2(\mathbb{Z}_N)$ if and only if
    1) $|\hat{u}(n)|^2 + |\hat{u}(n+M)|^2 = 2$
    2) $|\hat{v}(n)|^2 + |\hat{v}(n+M)|^2 = 2$
    3) $\hat{u}(n)\bar{\hat{v}}(n) + \hat{u}(n+M)\bar{\hat{v}}(n+M) = 0$ for all n = 0, 1, ..., M – 1.

39. Suppose $z \in l^2(\mathbb{Z})$ and $w \in l^1(\mathbb{Z})$. Then prove that $z * w \in l^2(\mathbb{Z})$ and $\|z * w\| \le \|w\|_1 \|z\|$.

40. Suppose l is a positive integer, $g_{l-1} \in l^2(\mathbb{Z})$ and $\{R_{2^{l-1}k}\, g_{l-1}\}_{k\in\mathbb{Z}}$ is orthonormal in $l^2(\mathbb{Z})$. Suppose also that $u, v \in l^1(\mathbb{Z})$ and the system matrix A(θ) of u and v is unitary for all θ. Define $f_l = g_{l-1} * U^{l-1}(v)$ and $g_l = g_{l-1} * U^{l-1}(u)$. Then $\{R_{2^l k} f_l\}_{k\in\mathbb{Z}} \cup \{R_{2^l k} g_l\}_{k\in\mathbb{Z}}$ is orthonormal.

41. Suppose $M \in \mathbb{N}$, N = 2M and $w \in l^2(\mathbb{Z}_N)$. Then $\{R_{2k}w\}_{k=0}^{M-1}$ is an orthonormal set with M elements if and only if $|\hat{w}(n)|^2 + |\hat{w}(n+M)|^2 = 2$ for n = 0, 1, ..., M – 1.

42. Suppose $w, z \in l^1(\mathbb{Z})$. Then prove that the set $\{R_{2k}w\}_{k\in\mathbb{Z}}$ is orthonormal if and only if $|\hat{w}(\theta)|^2 + |\hat{w}(\theta+\pi)|^2 = 2$ for all $\theta \in [0, \pi)$.

Reg. No. : ....

Name : ....

Sixth Semester B.Sc. Hon's. (Mathematics) Degree (Reg./Supple./Improv.) Examination, April 2021
(2016 Admission Onwards)
BHM604 – A : DISCRETE FOURIER ANALYSIS

Time : 3 Hours Max. Marks : 60

SECTION – A

(Answer **any 4** questions out of 8 questions. **Each** question carries **1** mark.) **(4×1=4)**

1. State Fourier inversion formula.
2. Describe first-stage wavelet basis for $l^2(\mathbb{Z}_N)$.
3. Define $L^2([-\pi, \pi))$ and $L^1([-\pi, \pi))$.
4. Define upsampling operator.
5. Define p^{th}-stage wavelet system for $l^2(\mathbb{Z})$.
6. When we say that a linear transformation from $l^2(\mathbb{Z})$ to $l^2(\mathbb{Z})$ is translation invariant ?
7. Define conjugate reflection of an element of $l^2(\mathbb{Z}_N)$.
8. For $u, v \in l^2(\mathbb{Z}_N)$ and $n \in \mathbb{Z}$, how will you define system matrix of u and v ?

SECTION – B

(Answer **any 6** questions out of 12 questions. **Each** question carries **2** marks.) **(6×2=12)**

9. Suppose $T : l^2(\mathbb{Z}_N) \to l^2(\mathbb{Z}_N)$ is a linear transformation. Let $A_{T,E}$ be the matrix representing T in the standard basis E. If T is translation invariant, then prove that $A_{T,E}$ is circulant.

10. Let z = (1, 1, 0, 2) and w = (i, 0, 1, i) be vectors in $l^2(\mathbb{Z}_4)$. Then find the vector $z * w$.

11. Let $\hat{u} = (\sqrt{2}, 1, 0, 1)$ and $\hat{v} = (0, 1, \sqrt{2}, 1)$. Then find A(0) and A(1), where A is the system matrix.

12. Prove that $\hat{v}(\theta+\pi) = -e^{i\theta}\overline{\hat{u}(\theta)}$.

13. Suppose N is divisible by 2^p. Suppose $u, v \in l^2(\mathbb{Z}_N)$ are such that the system matrix A(n) is unitary for all n. Let $u_1 = u$ and $v_1 = v$ and for all $l = 2, 3, \ldots, p$, define u_l by equation $u_l(n) = \sum_{k=0}^{2^{l-1}-1} u_1(n + kN/2^{l-1})$ and v_l similarly with v_1 in place of u_1. Then $u_1, v_1, u_2, v_2, \ldots, u_p, v_p$ is a p^{th}-stage wavelet filter sequence.

14. Prove that $L^2([-\pi, \pi))$ is vector space.

15. State and prove Plancherel's formula.

16. Suppose $M \in \mathbb{N}$ and $N = 2M$. Suppose $u \in l^1(\mathbb{Z})$. Then prove that $\hat{u}_{(N)}(m) = \hat{u}(-2\pi m / N)$.

17. Suppose $w \in l^1(\mathbb{Z})$. Then prove that $(w * \tilde{w}) + (w * \tilde{w})^* = 2\delta$.

18. Prove that the trigonometric system $\{e^{in\theta}\}_{n\in\mathbb{Z}}$ is an orthonormal set in $L^2([-\pi, \pi))$.

19. For any $w \in l^2(\mathbb{Z}_N)$, then prove that $w * \delta = w$.

20. Prove that trigonometric system is complete in $L^2([-\pi, \pi))$.

SECTION – C

(Answer **any 8** questions out of 16 questions. **Each** question carries 4 marks.) **(8×4=32)**

21. Let $b \in l^2(\mathbb{Z}_N)$ and T_b be the convolution operation associated with b. Then prove that T_b is translation invariant.

22. Suppose $z, w \in l^2(\mathbb{Z}_N)$. Then prove that for each m, $(z * w)^{\wedge}(m) = \hat{z}(m)\hat{w}(m)$.

23. Define $T : l^2(\mathbb{Z}_4) \to l^2(\mathbb{Z}_4)$ by $T(z)(n) = z(n) + 2z(n+1) + z(n+3)$. Find the eigenvalues and eigenvectors of T, and diagonalize the matrix A representing T in the standard basis, if possible.

24. Suppose $M \in \mathbb{N}$, $N = 2M$ and $u \in l^2(\mathbb{Z}_N)$ is such that $\{R_{2k}u\}_{k=0}^{M-1}$ is an orthonormal set with M elements. Define $v \in l^2(\mathbb{Z}_N)$ by $v(k) = (-1)^{k-1}\overline{u(1-k)}$ for all k. Then prove that $\{R_{2k}v\}_{k=0}^{M-1} \cup \{R_{2k}u\}_{k=0}^{M-1}$ is a first-stage wavelet basis for $l^2(\mathbb{Z}_N)$.

25. Suppose $f : [-\pi, \pi) \to \mathbb{C}$ is continuous and bounded, say $|f(\theta)| \le M$ for all θ. If $\langle f, e^{in\theta} \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, e^{-in\theta} d\theta$ for all $n \in \mathbb{Z}$. Then, prove that $f(\theta) = 0$ for all $\theta \in [-\pi, \pi)$.

26. Suppose N is divisible by 2 and $u_1 \in l^2(\mathbb{Z}_N)$. Define $u_2 \in l^2(\mathbb{Z}_{N/2})$ by $u_2(n) = u_1(n) + u_1(n + N/2)$. Then for all m, $\hat{u}_2(m) = \hat{u}_1(2m)$.

27. Prove that the Fourier transform $\hat{z}$ on $L^2([-\pi, \pi))$ is one-to-one and onto. Also prove that $\langle z, w \rangle = \langle \hat{z}, \hat{w} \rangle$ and $\|z\|^2 = \|\hat{z}\|^2$.

28. Let $p \in \mathbb{N}$. For $l = 1, 2, \ldots, p$, suppose that $u_l, v_l \in l^1(\mathbb{Z})$ and the system matrix $A_l(\theta)$ defined as

$$A_l(\theta) = \frac{1}{\sqrt{2}}\begin{bmatrix} \hat{u}_l(\theta) & \hat{v}_l(\theta) \\ \hat{u}_l(\theta+\pi) & \hat{v}_l(\theta+\pi) \end{bmatrix}$$ is unitary for all $\theta \in [0, \pi)$. Define $f_1 = v_1$, $g_1 = u_1$ and inductively, for $l = 2, 3, \ldots, p$, define f_l and g_l by $f_l = g_{l-1} * U^{l-1}(v_l)$, $g_l = g_{l-1} * U^{l-1}(u_l)$. Define $B = \{R_{2^l k} f_l : k \in \mathbb{Z}, l = 1, 2, \ldots, p\} \cup \{R_{2^p k} g_p : k \in \mathbb{Z}\}$. Then prove that B is a complete orthonormal set for $l^2(\mathbb{Z})$.

29. Suppose $M \in \mathbb{N}$ and $N = 2M$. Suppose $u, v \in l^1(\mathbb{Z})$ are such that $\{R_{2k}v\}_{k\in\mathbb{Z}} \cup \{R_{2k}u\}_{k\in\mathbb{Z}}$ is a first-stage wavelet system for $l^2(\mathbb{Z})$. Define $u_{(N)}, v_{(N)} \in l^2(\mathbb{Z}_N)$ by $u_{(N)}(n) = \sum_{k\in\mathbb{Z}} u(n + kN)$ and $v_{(N)}(n) = \sum_{k\in\mathbb{Z}} v(n + kN)$. Then prove that $\{R_{2k}v_{(N)}\}_{k=0}^{M-1} \cup \{R_{2k}u_{(N)}\}_{k=0}^{M-1}$ is a first-stage wavelet basis for $l^2(\mathbb{Z}_N)$.

30. Define $E_0, E_1, \ldots, E_{N-1} \in l^2(\mathbb{Z}_N)$ by $E_m(n) = \frac{1}{\sqrt{N}} e^{2\pi imn/N}$ for $0 \le m, n \le N-1$. Then the set $\{E_0, E_1, \ldots, E_{N-1}\}$ is an orthonormal basis for $l^2(\mathbb{Z}_N)$.

31. Suppose H is a Hilbert space and $T : H \to H$ is a bounded linear transformation. Suppose the series $\sum_{n\in\mathbb{Z}} x_n$ converges in H. Then prove that $T\left(\sum_{n\in\mathbb{Z}} x_n\right) = \sum_{n\in\mathbb{Z}} T(x_n)$ where the series on the right converges in H.

32. Prove that $l^2(\mathbb{Z})$ is complete.

33. Suppose $T : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ is a bounded, translation-invariant linear transformation. Define $b \in l^2(\mathbb{Z})$ by $b = T(\delta)$. Then prove that for all $z \in l^2(\mathbb{Z})$, $T(z) = b * z$.

34. Suppose $T : L^2([-\pi, \pi)) \to L^2([-\pi, \pi))$ is a bounded, translation-invariant linear transformation. Then prove that, for each $m \in \mathbb{Z}$, there exists $\lambda_m \in \mathbb{C}$ such that $T(e^{im\theta}) = \lambda_m e^{im\theta}$.