K18U 0324

Reg. No. : ..............................

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**VI Semester B.Sc. Hon's (Mathematics) Degree (Regular/Supple./Improv.) Examination, May 2018**
**(2013-15 Admns.)**
**BHM 605 (A) : DISCRETE FOURIER ANALYSIS**

Time : 3 Hours | Max. Marks : 80

Answer **all** the **ten** questions : **(10×1=10)**

1. Write Plancheral's formula in $\ell^2(\mathbb{Z}_N)$.
2. Define the translation by k operator $R_k$ in $\ell^2(\mathbb{Z}_N)$.
3. Define the conjugate reflection of an element $w \in \ell^2(\mathbb{Z}_N)$.
4. Define father wavelet.
5. Define the decimation operator from $\ell^2(\mathbb{Z}_N)$ to $\ell^2(\mathbb{Z}_M)$, where $M \in \mathbb{N}$, $N = 2M$.
6. Define absolute convergence of sequences.
7. Define the translation operator $\tau_\varphi$ for $\varphi \in \mathbb{R}$.
8. Suppose u, $v \in \ell^2(\mathbb{Z})$. What is the system matrix of u and v ?
9. Define p[th] stage wavelet system for $\ell^2(\mathbb{Z})$.
10. Define a homogeneous wavelet system for $\ell^2(\mathbb{Z})$.

Answer **any 10** short answer questions out of 14. **(10×3=30)**

11. Let $z = (1, 0, -3, 4) \in \ell^2(\mathbb{Z}_4)$. Find $\hat{z}$.
12. Find the discrete Fourier transform of $z(n) = \cos\left(2\pi \frac{11n}{128}\right) + 4\cos\left(2\pi \frac{28n}{128}\right)$ in $\ell^2(\mathbb{Z}_{128})$.
13. Define the Dirac delta function $\delta(n)$ and prove that $w * \delta = w$ for any $w \in \ell^2(\mathbb{Z}_N)$.

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14. Define a circulant matrix and show that $\begin{bmatrix} 2 & i & 3 \\ 3 & 2 & i \\ i & 3 & 2 \end{bmatrix}$ is circulant. Give an example of a matrix which is not circulant.

15. For $z \in \ell^2(\mathbb{Z}_N)$, define $T(z) \in \ell^2(\mathbb{Z}_N)$ by $T(z)(n) = z(n-1)\ \forall\ n$. Prove that T is translation invariant.

16. Suppose $z, w \in \ell^2(\mathbb{Z}_N)$. For any $k \in \mathbb{Z}$ show that $z * \tilde{w}(k) = \langle z, R_k w\rangle$ and $z * w(k) = \langle z, R_k \tilde{w}\rangle$, where $\tilde{w}$ is the conjugate reflection of $w \in \ell^2(\mathbb{Z}_N)$.

17. Explain upsampling operator from $\ell^2(\mathbb{Z}_M)$ to $\ell^2(\mathbb{Z}_N)$, where $M \in \mathbb{N}$, $N = 2M$.

18. Suppose $z, w \in \ell^2(\mathbb{Z}_N)$. Prove that $\langle R_k z, R_j w\rangle = \langle z, R_{j-k} w\rangle = \langle R_{k-j} z, w\rangle\ \forall k, j \in \mathbb{Z}$.

19. Suppose $N = 2M$, $x, y \in \ell^2(\mathbb{Z}_{N/2})$. Then prove that $U(x) * U(y) = U(x*y)$.

20. For $z, w \in \ell^2(\mathbb{Z})$, define $\langle z, w\rangle = \sum_{n\in\mathbb{Z}} z(n)\overline{w(n)}$. Show that $\langle .,.\rangle$ is an inner product on $\ell^2(\mathbb{Z})$.

21. State Fourier transform formula from $\ell^2(\mathbb{Z})$ to $L^2([-\pi, \pi))$ and the corresponding inverse Fourier transform formula.

22. Suppose $z \in \ell^2(\mathbb{Z})$ and $w \in \ell^1(\mathbb{Z})$. Then prove that $z * w \in \ell^2(\mathbb{Z})$ and $\|z*w\| \le \|w\|_1 \|z\|$.

23. Suppose $z \in \ell^2(\mathbb{Z})$. Show that $\widehat{z^*}(\theta) = \hat{z}(\theta + \pi)$.

24. Explain the process of obtaining a periodized wavelets for $\ell^2(\mathbb{Z}_N)$.

Answer **any 6** short answer questions out of 9. **(6×5=30)**

25. Prove that the set $\{E_0, E_1, \ldots, E_{N-1}\}$ is an orthonormal basis for $\ell^2(\mathbb{Z}_N)$, where $E_0(n) = \frac{1}{\sqrt{N}}$ for $n = 0, 1, 2, \ldots, N-1$ and $E_m(n) = \frac{1}{\sqrt{N}} e^{\frac{2\pi i mn}{N}}$ $0 \le m, n \le N-1$.
Using this information, find a basis for $\ell^2(\mathbb{Z}_4)$.

26. What do you mean by convolution operator in $\ell^2(\mathbb{Z}_N)$ ? Prove that the convolution operator is translation invariant.



27. Let $\hat{u} = (\sqrt{2}, \sqrt{2}, 0, 0)$ and $\hat{v} = (0, 0, \sqrt{2}, \sqrt{2})$. Find $\{u, R_2 u, v, R_2 v\}$ for $\ell^2(\mathbb{Z}_4)$.
Show that the above set is an orthonormal basis.

28. State and prove folding lemma.

29. Suppose $f \in L^1([-\pi, \pi))$ and $\langle f, e^{in\theta}\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta = 0$ for all $n \in \mathbb{Z}$. Prove that $f(\theta) = 0$ a.e.

30. Suppose $T: L^2([-\pi, \pi)) \to L^2([-\pi, \pi))$ is a bounded, translation-invariant linear transformation. Then prove that for each $m \in \mathbb{Z}$, there exists, $\lambda_m \in \mathbb{C}$ such that $T(e^{im\theta}) = \lambda_m e^{im\theta}$.

31. Suppose $T: \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is a bounded, translation-invariant linear transformation. Define $b \in \ell^2(\mathbb{Z})$ by $b = T(\delta)$, where $\delta$ is the delta function. Then prove that for all $z \in \ell^2(\mathbb{Z})$, $T(z) = b * z$.

32. Suppose $v, u \in \ell^1(\mathbb{Z})$. Prove that $B = \{R_{2k}v\}_{k\in\mathbb{Z}} \cup \{R_{2k}u\}_{k\in\mathbb{Z}}$ is a complete orthonormal set in $\ell^2(\mathbb{Z})$ if and only if the system matrix $A(\theta)$ is unitary for all $\theta \in [0, \pi)$.

33. For a sequence $z \in \ell^2(\mathbb{Z})$, define the downsampling operator and the upsampling operator. Also prove that $D \circ U(z) = z$ and $U \circ D(z) = \frac{1}{2}(z + z^*)$.

Answer **any one** essay question out of 2. **(1×10=10)**

34. Explain the construction of Daubechies's $D_6$ wavelets on $\mathbb{Z}_N$, briefly.

35. i) Define the orthogonal direct sum of two subspaces U and V of an inner product space X and give an example. **2**

ii) Suppose $l$ is a positive integer, $g_{l-1} \in \ell^2(\mathbb{Z})$ and $\{R_{2^{l-1}k} g_{l-1}\}_{k\in\mathbb{Z}}$ is orthonormal in $\ell^2(\mathbb{Z})$. Suppose also that $u, v \in \ell^1(\mathbb{Z})$ and the system matrix $A(\theta)$ of u and v is unitary for all $\theta$. Define $f_l = g_{l-1} * U^{l-1}(v)$ and $g_l = g_{l-1} * U^{l-1}(u)$. Define :
$V_{-l+1} = \left\{\sum_{k\in\mathbb{Z}} z(k) R_{2^{l-1}k}\, g_{l-1} : z = (z(k))_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})\right\}$
$V_{-l} = \left\{\sum_{k\in\mathbb{Z}} z(k) R_{2^l k}\, g_l : z = (z(k))_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})\right\}$ and
$W_{-l} = \left\{\sum_{k\in\mathbb{Z}} z(k) R_{2^l k}\, f_l : z = (z(k))_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})\right\}$. Then prove that $V_{-l} \oplus W_{-l} = V_{-l+1}$. **8**

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