26. Suppose $u_l, v_l \in l^1(\mathbb{Z})$ for each $l \in N$ and the system matrix $A_l(\theta)$ defined as

$$A_l(\theta) = \frac{1}{\sqrt{2}}\begin{bmatrix} \hat{u}_l(\theta) & \hat{v}_l(\theta) \\ \hat{u}_l(\theta+\pi) & \hat{v}_l(\theta+\pi) \end{bmatrix}$$

is unitary for all $\theta \in [0, \pi)$. Define $f_1 = u_1$, $g_1 = v_1$ and inductively for $l \in N$, $l \geq 2$, define $f_l$ and $g_l$ by $f_l = g_{l-1} * U^{l-1}(v_l)$, $g_l = g_{l-1} * U^{l-1}(u_l)$. For each $l \in N$, define $V_{-l}$ as $V_{-l} = \left\{\sum_{k\in\mathbb{Z}} z(k) R_{2^l k} g_l : z = (z(k))_{k\in\mathbb{Z}} \in l^2(\mathbb{Z})\right\}$. Suppose $\bigcap_{l\in N} V_{-l} = \{0\}$ and B is defined as $B = \{R_{2^l k} f_l : k \in \mathbb{Z}, l \in N\}$. Then prove that B is a complete orthonormal set in $l^2(\mathbb{Z})$.

SECTION – D

Answer **any 2** questions out of 4 questions. **Each** question carries **6** marks : **(2×6=12)**

27. Prove that : $T : l^2(\mathbb{Z}_N) \to l^2(\mathbb{Z}_N)$ be a linear transformation. Then T is a Fourier multiplier operator $T_m$ for some $m \in l^2(\mathbb{Z}_N)$ if and only if the matrix representing T in the Fourier basis $F = \{F_0, F_1, \ldots, F_{N-1}\}$ is a diagonal matrix D. Moreover, if $T = T_m$ is a Fourier multiplier operator, then the diagonal matrix $D = [d_{mn}]$, $0 \leq m, n \leq N-1$ satisfies $d_{nn} = m(n)$ for $n = 0, 1, \ldots, N-1$.

28. Suppose $N = 2^n$, $1 \leq p \leq n$ and $u_1, v_1, u_2, v_2, \ldots, u_p, v_p$ form a $p^{th}$ stage wavelet filter sequence. Suppose $z \in l^2(\mathbb{Z}_N)$. Then prove that the output $\{x_1, x_2, x_3, \ldots, x_p, y_p\}$ of the analysis phase of the corresponding $p^{th}$ stage wavelet filter bank can be computed using no more than $4N + N\log_2 N$ complex multiplications.

29. Suppose $v, w \in l^1(\mathbb{Z})$ and $z \in l^2(\mathbb{Z})$. Then prove that
   a) $(z * w)^\wedge(\theta) = \hat{z}(\theta)\hat{w}(\theta)$ a.e.
   b) $z * w = w * z$.
   c) $v*(w * z) = (v * w) * z$.

30. Suppose that $u, v \in l^1(\mathbb{Z})$. Then prove that $B = \{R_{2k} v\}_{k\in\mathbb{Z}} \cup \{R_{2k} u\}_{k\in\mathbb{Z}}$ is a complete orthonormal set in $l^2(\mathbb{Z})$ if and only if the system matrix $A(\theta)$ is unitary for all $\theta \in [0, \pi)$.

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Reg. No. : ...............................

Name : ...................................

**VI Semester B.Sc. Hon's (Mathematics) Degree (Reg./Supple./Improv.) Examination, April 2020**
**(2016 Admission Onwards)**
**BHM604A : DISCRETE FOURIER ANALYSIS**

Time : 3 Hours Max. Marks : 60

SECTION – A

Answer **any 4** questions out of 5 questions. **Each** question carries **1** mark : **(4×1=4)**

1. Let $E_0 = (E_0(0), E_0(1)) = \frac{1}{\sqrt{2}}(1,1)$ and $E_1 = (E_1(0), E_1(1)) = \frac{1}{\sqrt{2}}(1,-1)$. Then prove that $\{E_0, E_1\}$ is an orthonormal basis for $l^2(\mathbb{Z}_2)$.

2. Define $p^{th}$ stage wavelet filter sequence.

3. For $u, v \in l^2(\mathbb{Z}_N)$ and $n \in \mathbb{Z}$, how will you define system matrix of u and v ?

4. Define a translation operator on $L^2([-\pi, \pi,))$.

5. Define downsampling operator and upsampling operator on $l^2(\mathbb{Z})$.

SECTION – B

Answer **any 6** questions out of 9 questions. **Each** question carries **2** marks : **(6×2=12)**

6. Prove that for $z \in l^2(\mathbb{Z}_N)$, $(\overline{z})^\wedge(m) = \overline{\hat{z}(-m)} = \overline{\hat{z}(N-m)}$, for all m.

7. Let A be an $N \times N$ matrix, $A = [a_{m,n}]$, $0 \leq m, n \leq N-1$. Suppose A is circulant. Define $b \in l^2(\mathbb{Z}_N)$ by $b(n) = a_{n,0}$ for all n. Then prove that for all $z \in l^2(\mathbb{Z}_N)$, $Az = b * z = T_b(z)$.

P.T.O.

8. Suppose $z, w \in l^2(\mathbb{Z}_N)$. Then prove for any $k \in \mathbb{Z}$, that $z * \tilde{w}(k) = \langle z, R_k w \rangle$ and $z * w(k) = \langle z, R_k \tilde{w} \rangle$.

9. Suppose $M \in \mathbb{N}$, $N = 2M$ and $z \in l^2(\mathbb{Z}_N)$. Define $z^* \in l^2(\mathbb{Z}_N)$ by $z^*(n) = (-1)^n z(n)$ for all n. Then prove that $(z^*)\,\hat{}\,(n) = \hat{z}(n + M)$ for all n.

10. Suppose $z = (2, 5, -1, i) \in l^2(\mathbb{Z}_N)$. Give example of a upsampling operator U(z).

11. Prove that triangle inequality holds in $L^2([-\pi, \pi))$.

12. State and prove Parseval's relation.

13. Suppose $u, v \in l^1(\mathbb{Z})$ are such that $\{R_{2k}v\}_{k\in\mathbb{Z}} \cup \{R_{2k}u\}_{k\in\mathbb{Z}}$ is a first stage wavelet system for $l^2(\mathbb{Z})$. Suppose also that $u(n) = v(n) = 0 \ \forall \ n < 0$ and $\forall n > N - 1$. Define $u_{(N)}, v_{(N)} \in l^2(\mathbb{Z}_N)$ by $u_{(N)}(n) = u(n)$ and $v_{(N)}(n) = v(n)$ for $n = 0, 1, 2, ..., N - 1$. Then prove that $\{R_{2k}v_{(N)}\}_{k=0}^{N-1} \cup \{R_{2k}u_{(N)}\}_{k=0}^{M-1}$ is a first-stage wavelet basis for $l^2(\mathbb{Z}_N)$.

14. Prove that $\hat{v}(\theta + \pi) = -e^{i\theta}\overline{\hat{u}(\theta)}$.

## SECTION – C

Answer **any 8** questions out of 12 questions. **Each** question carries **4** marks : **(8×4=32)**

15. Define $E_0, E_1, ..., E_{N-1} \in l^2(\mathbb{Z}_N)$ by $E_m(n) = \frac{1}{\sqrt{N}} e^{2\pi i mn/N}$ for $0 \le m, n \le N - 1$. Then the set $\{E_0, E_1, ..., E_{N-1}\}$ is an orthonormal basis for $l^2(\mathbb{Z}_N)$.

16. Let $b \in l^2(\mathbb{Z}_N)$ and $T_b$ be the convolution operation associated with b. Then prove that $T_b$ is translation invariant.

17. Prove that $T : l^2(\mathbb{Z}_N) \to l^2(\mathbb{Z}_N)$ be a linear transformation. Then T is a convolution operator if and only if T is a Fourier multiplier operator.

18. Let $w \in l^2(\mathbb{Z}_N)$. Then prove that $\{R_k w\}_{k=0}^{N-1}$ is an orthonormal basis for $l^2(\mathbb{Z}_N)$ if and only if $|\hat{w}(n)| = 1 \ \forall n \in \mathbb{Z}_N$.



19. Suppose N is even, say N = 2M, $z \in l^2(\mathbb{Z}_N)$ and x, y, $w \in l^2(\mathbb{Z}_{N/2})$. Then prove that

    $D(z) * w = D(z * U(w))$ and

    $U(x) * U(y) = U(x * y)$.

20. Suppose N is divisible by $2^p$ and $u_1, v_1, u_2, v_2, ..., u_p, v_p$ is a $p^{th}$-stage wavelet filter sequence. Define $f_1, f_2, ..., f_p, g_1, g_2, ..., g_p$ inductively by $f_1 = v_1, g_1 = u_1$, $f_l = g_{l-1} * U^{l-1}(v_l), g_l = g_{l-1} * U^{l-1}(u_l)$. Then prove that $f_1, f_2, ..., f_p, g_p$ generate a $p^{th}$-stage wavelet basis for $l^2(\mathbb{Z}_N)$.

21. Suppose $\theta_0 \in (-\pi, \pi)$ and $\alpha > 0$ is sufficiently small that $-\pi < \theta_0 - \alpha < \theta_0 + \alpha < \pi$. Define intervals $I = (\theta_0 - \alpha, \theta_0 + \alpha)$ and $J = (\theta_0 - \alpha/2, \theta_0 + \alpha/2)$. Then prove that there exists $\delta > 0$ and a sequence of real-valued trigonometric polynomials $\{p_n(\theta)\}_{n=1}^{\infty}$ such that

    a) $p_n(\theta) \ge 1, \forall \theta \in I$

    b) $p_n(\theta) \ge (1 + \delta)^n, \forall \theta \in J$

    c) $|p_n(\theta)| \le 1, \forall \theta \in [-\pi, \pi) \setminus I$.

22. Suppose $f \in L^1([-\pi, \pi))$ and $\langle f, e^{in\theta} \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) e^{-in\theta}\, d\theta = 0, \forall n \in \mathbb{Z}$. Then prove that $f(\theta) = 0$ a.e.

23. Suppose $z, w \in l^2(\mathbb{Z})$. Then prove that

    a) $\tilde{z}, z^* \in l^2(\mathbb{Z})$ and $R_k z \in l^2(\mathbb{Z})$, for all $k \in \mathbb{Z}$.

    b) $(\tilde{z})^{\wedge}(\theta) = \overline{\hat{z}(\theta)}$.

    c) $(z^*)^{\wedge}(\theta) = \hat{z}(\theta + \pi)$.

24. Suppose $u \in l^1(\mathbb{Z})$ and $\{R_{2k}u\}_{k\in\mathbb{Z}}$ is orthonormal in $l^2(\mathbb{Z})$. Define a sequence $v \in l^1(\mathbb{Z})$ by $v(k) = (-1)^{k-1}\overline{u(1-k)}$. Then prove that $\{R_{2k}v\}_{k\in\mathbb{Z}} \cup \{R_{2k}u\}_{k\in\mathbb{Z}}$ is a complete orthonormal system in $l^2(\mathbb{Z})$.

25. Suppose $M \in \mathbb{N}$ and N = 2M. Suppose $u, v \in l^1(\mathbb{Z})$ are such that $\{R_{2k}v\}_{k\in\mathbb{Z}} \cup \{R_{2k}u\}_{k\in\mathbb{Z}}$ is a first-stage wavelet system for $l^2(\mathbb{Z})$. Define $u_{(N)}, v_{(N)} \in l^2(\mathbb{Z}_N)$ by $u_{(N)}(n) = \sum_{k\in\mathbb{Z}} u(n + kN)$ and $v_{(N)}n = \sum_{k\in\mathbb{Z}} v(n + kN)$. Then prove that $\{R_{2k}v_{(N)}\}_{k=0}^{M-1} \cup \{R_{2k}u_{(N)}\}_{k=0}^{M-1}$ is a first-stage wavelet basis for $l^2(\mathbb{Z}_N)$.