K19U 0764

Reg. No. : ....

Name : ....

VI Semester B.Sc. Hon's (Mathematics) Degree (Reg.) Examination, April 2019
(2016 Admission)
BHM604A : DISCRETE FOURIER ANALYSIS

Time : 3 Hours Max. Marks : 60

PART – A

Answer **any four** questions. **Each** question carries **one** mark.

1. Define inverse discrete Fourier transform.
2. Define p^{th} stage wavelet basis for $l^2(\mathbb{Z}_N)$.
3. Show that $(\tilde{\omega})^\wedge(n) = \overline{\omega^\wedge(n)}$ on $l^2(\mathbb{Z}_N)$.
4. Define inverse Fourier transform on $L^2([-\pi, \pi))$.
5. Suppose $z, \omega \in l^1(\mathbb{Z})$. State the condition for which the set $\{R_{2k}\omega\}_{k \in \mathbb{Z}}$ is orthonormal.

PART – B

Answer **any six** questions. **Each** question carries **two** marks.

6. If $z = (1, 1, 0, 2)$, $\omega = (1, 0, 1, i)$ find $z*\omega(0)$ and $z*\omega(1)$.
7. Prove that for any $\omega \in l^2(\mathbb{Z}_N)$, $\omega * \delta = \omega$ where δ is the dirac delta function.
8. Suppose $M \in \mathbb{N}$ $N = 2M$ $z \in l^2(\mathbb{Z}_N)$ define $z^* \in l^2(\mathbb{Z}_N)$ by $z^*(n) = (-1)^n z(n)$, $\forall n$ then show that $\hat{z^*}(n) = \hat{z}(n+M)$.
9. Suppose $z, \omega \in l^2(\mathbb{Z}_N)$, then prove that $\widetilde{(z * \omega)} = \tilde{z} * \tilde{\omega}$.
10. Suppose $z, \omega \in l^2(\mathbb{Z})$ prove that $(z^*)^\wedge(\theta) = \hat{z}(\theta + \pi)$.
11. Suppose $T : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ is a bounded translation invariant linear transformation. Define $b \in l^2(\mathbb{Z})$ by $b = T(\delta)$. Show that $\forall z \in l^2(\mathbb{Z})$, $T(z) = b * z$.

P.T.O.

12. Write a note on $l^2(\mathbb{Z})$, norm and inner product on $l^2(\mathbb{Z})$.

13. Define the m-fold composition of up sampling and down sampling operator in $l^2(\mathbb{Z})$.

14. Suppose $z \in l^2(\mathbb{Z})$. Prove that $(U(\hat{z}))(\theta) = \hat{z}(2\theta)$.

PART – C

Answer **any eight** questions. **Each** question carries **four** marks.

15. If $\omega = (2, 4 + 4i, -6, 4 - 4i)$ find $\hat{\omega}$.

16. Suppose $z, \omega \in l^2(\mathbb{Z}_N)$, for each m, show that $(z * \omega)^\wedge(m) = \hat{z}(m) * \hat{\omega}(m)$.

17. Suppose $\Delta : l^2(Z_N) \to l^2(Z_N)$ is defined by $\Delta(z)(n) = z(n+1) - 2z(n) + z(n-1)$. Find the eigen values of Δ.

18. Suppose $\hat{u} = \left(\sqrt{2}, \sqrt{2}, 0, 0\right)$, $\hat{v} = \left(0, 0, \sqrt{2}, \sqrt{2}\right)$ check whether the system matrix A (n) is unitary. Deduce that $\{u, R_2u, v, R_2v\}$ is an orthonormal basis for $l^2(\mathbb{Z}_4)$.

19. Let $u \in l^2(\mathbb{Z}_4)$ such that $\hat{u} = \left(1, \sqrt{2}, i, 0\right)$. Find some $\hat{v}$ such that $\{u, R_2u, v, R_2v\}$ is an orthonormal basis for $l^2(\mathbb{Z}_4)$.

20. Write a note on first stage Shannon basis.

21. Suppose H is a Hilbert space, $S = \left\{ \sum_{j \in \mathbb{Z}} z(j) a_j : z = (z(j))_{j \in \mathbb{Z}} \in l^2(\mathbb{Z}) \right\}$ and $P_S(f) = \sum_{j \in \mathbb{Z}} \langle f, a_j \rangle a_j$ then, show that

    a) $\langle f - P_s(f)\ s \rangle = 0$ for any $f \in H$ and $s \in S$.

    b) $\| f - P_s(f) \| \leq \| f - s \|$ with equality only for $s = P_s(f)$.

22. Suppose H is a Hilbert space $\{a_j\}_{j \in \mathbb{Z}}$ is an orthonormal set in H and $f \in H$. Show that the sequence $\left\{\langle f, a_j \rangle\right\}_{j \in \mathbb{Z}}$ belongs to $l^2(\mathbb{Z})$ with $\sum_{j \in \mathbb{Z}} \left| \langle f, a_j \rangle \right|^2 \leq \| f \|^2$.

23. Suppose $N = 2^n$, $1 \leq p \leq n$ and $u_1, u_2, \ldots u_p, v_1, v_2, \ldots v_p$. Form a p^{th} stage wavelet filter sequence. Suppose $z \in l^2(\mathbb{Z}_N)$. Show that the output $\{x_1, x_2, x_3, \ldots x_p, y_p\}$ of the analysis phase of the corresponding p^{th} stage wavelet filter bank can be computed using no more than $4N + N \log_2 N$ complex multiplications.

24. Suppose $u \in l^1(\mathbb{Z})$ are such that $\{R_{2k} u\}_{k \in \mathbb{Z}}$ is orthonormal in $l^2(\mathbb{Z})$. Define a sequence $v \in l^1(\mathbb{Z})$ by $v(k) = (-1)^{k-1} \overline{u(1-k)}$. Show that $\{R_{2k}u\}_{k \in \mathbb{Z}} \cup \{R_{2k}u\}_{k \in \mathbb{Z}}$ is a complete orthonormal system in $l^2(\mathbb{Z})$.

25. Suppose l is a positive integer $g_{l-1} \in l^2(\mathbb{Z})$ and $\left\{R_{2^{l-1}k}\, g_{l-1}\right\}_{k \in \mathbb{Z}}$ is orthonormal in $l^2(\mathbb{Z})$. Suppose also that $u, v \in l^1(\mathbb{Z})$ and the system matrix $A(\theta)$ of u and v is unitary for all θ $f_l = g_{l-1} U^{l-1}(v)$ and $g_l = g_{l-1} U^{l-1}(u)$. Show that $\left\{R_{2^l k} f_l\right\}_{k \in \mathbb{Z}} \cup \left\{R_{2^l k} g_l\right\}_{k \in \mathbb{Z}}$ is orthonormal.

26. Suppose $z, \omega \in l^2(\mathbb{Z})$. Prove that $U(\omega * z) = U(\omega) * U(z)$ and $\widetilde{(z * \omega)} = \tilde{z} * \tilde{\omega}$.

PART – D

Answer **any two** questions. **Each** question carries **six** marks.

27. Let $T : l^2(Z_N) \to l^2(Z_N)$ be a translation invariant linear transformation. Show that each element of the Fourier basis F is an eigen vector of T and T is diagonalizable.

28. State and prove folding lemma.

29. State and prove Plancherel's formula for $L^2([-\pi, \pi))$.

30. Suppose $u_l, v_l \in l^1(\mathbb{Z})$ for each $l \in \mathbb{N}$ and the system matrix $A_l(\theta)$ is unitary for all $\theta \in [0, \pi)$. Define $f_1 = u_1$, $g_1 = v_1$ and for $l \in \mathbb{N}, l \geq 2$, $f_l = g_{l-1}U^{l-1}(v_l)$, $g_l = g_{l-1}U^{l-1}(u_l)$. For each $l \in \mathbb{N}$, consider $V_{-l} = \left\{ \sum_{k \in \mathbb{Z}} z(k) R_{2^l k} g_l : (z(k))_{k \in \mathbb{Z}} \in l^2(\mathbb{Z}) \right\}$. Suppose $\bigcap_{l \in \mathbb{N}} V_{-l} = \{0\}$. Show that $B = \left\{ R_{2^l k} f_l : k \in \mathbb{Z}, l \in \mathbb{N} \right\}$ is a complete orthonormal set in $l^2(\mathbb{Z})$.