Reg. No. : ....

Name : ....

V Semester B.Sc. Hon's (Mathematics) Degree (Reg./Supple./Improv.) Examination, November 2020
(2016 Admission Onwards)
BHM 504 : DIFFERENTIAL GEOMETRY

Time : 3 Hours Max. Marks : 60

SECTION – A

Answer **any 4** questions out of 5 questions. **Each** question carries **1** mark. **(4×1=4)**

1. Define the graph of a function $f : U \to \mathbb{R}$, $U \subseteq \mathbb{R}^{n+1}$.
2. If $f(x_1, \ldots, x_{n+1}) = x^2_1 + \ldots + x^2_{n+1}$, what is the level set $f^{-1}(c)$, for $c < 0$.
3. Define an integral curve.
4. Give an example of a surface on which there exists no orientation.
5. Define the covariant derivative of the tangent vector field $\mathbb{X}$ with respect to $\overline{v} \in S_p$.

SECTION – B

Answer **any 6** questions out of 9 questions. **Each** question carries **2** marks. **(6×2=12)**

6. Define a vector field and give an example.
7. Define the velocity vector at time t of the parametrized curve $\alpha : I \to \mathbb{R}^{n+1}$. If α is an integral curve of the vector field $\mathbb{X}$, give the relation between $\alpha(t)$ and $\mathbb{X}(\alpha(t))$.
8. Define a right handed and left handed ordered orthonormal basis for the tangent space S_p to S, where S is an oriented 3-surface in $\mathbb{R}^4$.
9. Define a smooth vector field $\mathbb{X}$ along a parametrized curve α and its derivative $\dot{\mathbb{X}}$.
10. For each pair of orthogonal unit vectors $\{e_1, e_2\}$ in $\mathbb{R}^3$ and each $a \in \mathbb{R}$, prove that the great circle $\alpha(t) = (\cos at)G + (\sin at)\, e_2$ is a geodesic in the 2-sphere $x^2_1 + x^2_2 + x^2_3 = 1$ in $\mathbb{R}^3$.

11. Show that the function $\bar{v}$ into $\nabla_{\bar{v}} f$ is a linear map from $\mathbb{R}_p^{n+1}$ to $\mathbb{R}$, where $f : U \to \mathbb{R}$, U is open in $\mathbb{R}^{n+1}$.

12. Why the Weingarten map L_p is sometimes called the shape operator of a surface at 'p' ?

13. Find $\nabla_{\bar{v}} f$ where $f(x_1, x_2, x_3) = x_1 x_2 x_3^2$ and $\bar{v} = (1, 1, 1, a, b, c)$.

14. Prove that local parametrization of plane curves are unique up to reparametrization.

SECTION – C

Answer **any 8** questions out of 12 questions. **Each** question carries **4** marks. **(8×4=32)**

15. Show that the gradient of a function $f : U \to \mathbb{R}$, $U \subseteq \mathbb{R}^{n+1}$, is orthogonal to all vectors tangent to $f^{-1}(c)$ at p.

16. Prove that the n-plane $a_1 x_1 + \ldots + a_{n+1} x_{n+1} = b$ is an n-surface, where $(a_1, \ldots, a_{n+1}) \in \mathbb{R}^{n+1}$ and $(a_1, \ldots, a_n) \neq 0$.

17. If S is a connected n-surface in $\mathbb{R}^{n+1}$, then show that there exists exactly two normal vector fields $\mathbb{N}_1$ and $\mathbb{N}_2$ on S such that $\mathbb{N}_2(p) = -\mathbb{N}_1(p)$ for all $p \in S$.

18. Prove that geodesics have constant speed. If an n-surfaces s contains a straight line segment, $\alpha(t) = p + tv$, where $\alpha : I \to S$ and $t \in I$, show that this segment is a geodesic in s.

19. Show that the great circle $\alpha(t) = \cos(at)\, e_1 + \sin(at)\, e_2$, where e_1 and e_2 are pair of orthogonal vectors in $\mathbb{R}^3$, is a geodesic in the 2-sphere $x_1^2 + x_2^2 + x_3^2 = 1$.

20. If $\mathbb{X}$ and $\mathbb{Y}$ are smooth vector fields along the parametrized curve $\alpha : I \to \mathbb{R}^{n+1}$, prove that $(\mathbb{X}\cdot\mathbb{Y})' = \dot{\mathbb{X}}.Y + \mathbb{X}\cdot\dot{\mathbb{Y}}$, where $\dot{\mathbb{X}}$ represents the derivative of $\mathbb{X}$ along α.

21. Let S be an n-surface in $\mathbb{R}^{n+1}$, let $p, q \in S$ and α be a piecewise smooth parametrized curve from p to q. Then prove that the parallel transport $P_\alpha : S_p \to S_q$ along α is a vectorspace isomorphism which preserves dot products.

22. Define Levi-Civita parallelism. Also, prove that if $\mathbb{X}$ and $\mathbb{Y}$ are parallel along α, then $\mathbb{X}$ has a constant length and $\mathbb{X}.\mathbb{Y}$ is constant along α.

23. Compute $\nabla_{\bar{v}} \mathbb{X}$, where $\bar{v} \in \mathbb{R}_p^{n+1}$ and $\mathbb{X}$ is given by $\mathbb{X}(x_1, x_2) = (x_1, x_2, x_1 x_2, x_2^2)$, $v = (1, 0, 0, 1)$ and $n = 1$.

24. Show that the Weingarten map of radius r is simply multiplication by $\frac{1}{r}$.

25. Define a local parametrization and global parametrization of a plane curve c.

26. Explain a method to obtain local parametrization of plane curves.

SECTION – D

Answer **any 2** questions out of 4. **Each** question carries **6** marks. **(2×6=12)**

27. If U is an open set in $\mathbb{R}^{n+1}$, $F : U \to \mathbb{R}$ is smooth, $P \in U$ is a regular point of f and $c = f(p)$, then show that the set of all vectors tangent to $f^{-1}(c)$ at p is equal to $[\nabla f(p)]^{\perp}$.

28. Let S be an n-surface in $\mathbb{R}^{n+1}$, $p \in S$ and $\bar{v} \in S_p$. Prove that there exists an open internal containing 0 and a geodesic $\alpha : I \to S$ such that $\alpha(0) = p$ and $\alpha(0) = \bar{v}$.

29. Let $f(x_1, x_2) = (x_1 - a)^2 + (x_2 - b)^2$ and let c be the circle $f^{-1}(r^2)$ oriented by the outward normal $\nabla f / \|\nabla f\|$. Prove that $K(\alpha(t)) = -\frac{1}{r}$, where $\alpha(t) = (a + r\cos 2t, b - r\sin 2t)$ is the global parametrization of c.

30. Prove that $L_p(\bar{v}).\bar{w} = \bar{v}.L_p(\bar{w})$, where L_p is the Weingarten map and $\bar{v}, \bar{w} \in S_p$.
