28. Let S be an n-surface in $\mathbb{R}^{n+1}$, let X be a smooth tangent vector field on S, and let $p \in S$. Then prove that there exists an open interval I containing O and a parametrized curve $\alpha : I \to S$ such that i) $\alpha(0) = p$, ii) $\dot{\alpha}(t) = X(\alpha(t))$ for all $t \in I$ and iii) If $\beta : \tilde{I} \to S$ is any other parametrized curve in S satisfying i) and ii), prove that $\tilde{I} \subset I$ and $\beta(t) = \alpha(t)$ for all $t \in \tilde{I}$

29. Let S be an n-surface in $\mathbb{R}^{n+1}$, $\alpha : I \to S$ be a parametrized curve in S to $\in I$, and let $\bar{v} \in S_{\alpha(t_0)}$. Then prove that there exists a unique vector field V, tangent to S along α, which is parallel and has $V(t_0) = \bar{v}$.

30. Prove that the Weingarten map L_p is self-adjoint

K19U 3028

Reg. No. : ....

Name : ....

V Semester B.Sc. Hon's(Mathematics) Degree (Reg./Supple./Improv.) Examination, November- 2019

(2016 Admission Onwards)

BHM 504 : DIFFERENTIAL GEOMETRY

Time : 3 Hours Max. Marks : 60

SECTION - A

Answer any **4** questions out of **5** questions. Each question carries **one** mark. **(4×1=4)**

1. Define the height of a level set.

2. What is the level set of the function-

   $f(x_1,...x_{n+1}) = x_1^2 + ... + x_n^2$ at c, where c>0, n=1.

3. Define the gradient of a smooth function

   $f : U \to \mathbb{R}, U \subseteq \mathbb{R}^{n+1}$ at a point $p \in U$.

4. Define a smooth vector field.

5. If X is a smooth vector field on an open set $U \subseteq \mathbb{R}^{n+1}$, define the derivative of X with respect to a vector $\bar{v} \in \mathbb{R}_p^{n+1}$, $p \in U$.

SECTION- B

Answer any **6** questions out of **9** questions. Each question carries **2** marks. **(6×2=12)**

6. Sketch the level sets and graph of $f : \mathbb{R}^2 \to \mathbb{R}$, where $f(x_1, x_2) = -x_1^2 + x_2^2$.

7. Define a vector at a point $p \in \mathbb{R}^{n+1}$ and interpret it geometrically.

8. Explain why the Mobius band is not a 2 - Surface.

9. Define the covariant derivative of a smooth vector field and prove that it is independent of the choice of $\mathbb{N}$, where $\mathbb{N}$ is the orientation-vector field.

10. If X and Y are parallel vector fields along α, prove that X.Y is constant along α

11. Show that the function which sends $\bar{v}$ to $\nabla_{\bar{v}} f$ is a linear map from $\mathbb{R}_p^{n+1}$ to $\mathbb{R}$.

12. Define the Weingarten map and interpret it geometrically.

13. Compute $\nabla_{\bar{v}} f$, where $f(x_1,x_2)=x_1^2-x_2^2$ where $\bar{v}=(1,0,2,1)$

14. Find $\nabla_{\bar{v}} X$, where $X(x_1, x_2)=(x_1, x_2, x_1 x_2, x_2^2)$ and $\bar{v}=(1,0,0,1)$

SECTION- C

Answer any **8** questions out of **12** questions. Each question carries **4** marks. **(8×4=32)**

15. Obtain the integral curve through (1, 0) and through an arbitrary point (a, b) of the the vector field X (p) = (p,X(p)) where - $X(x_1, x_2) = (-x_2, x_1)$.

16. Prove that the n-plane $a_1x_1 + \ldots + a_{n+1}x_{n+1} = b$ is an n-surface, where $(a_1, a_2, \ldots a_{n+1}) \in \mathbb{R}^{n+1}$ and $(a_1, \ldots, a_{n+1}) \neq 0$.

17. Let S be an n-surface in $\mathbb{R}^{n+1}$, $S=f^{-1}(c)$, where $f: U \to \mathbb{R}$ is such that $\nabla f(q) \neq 0$ for all $q \in S$ and $g: U \to \mathbb{R}$ is a smooth function and $p \in S$ is an extreme point of g on S. Prove that there exists a real number λ such that $\nabla g(p) = \lambda \nabla f(p)$.

18. Prove that each n-surface in $\mathbb{R}^{n+1}$ has exactly two orientations.

19. For each $a,b,c,d, \in \mathbb{R}$, Prove that the parametrized curve $\alpha(t)=(\cos(at+b), \sin(at+b), ct+d)$ is a geodesic in the cylinder $x_1^2+x_2^2=1$ in $\mathbb{R}^3$.

20. If X and Y are smooth vector fields along the parametrized curve α and f is a smooth function along α, prove that $\left(f\dot{X}\right)=f'X+f\dot{X}$.

21. If X and Y are smooth vector fields tangent to S along a parametrized curve $\alpha: I \to S$, show that $(X.Y)' = X'.Y + X.Y'$, where X is the covariant derivative of X.

22. For $\theta \in \mathbb{R}$, let $\alpha_\theta : [0,\pi] \to S^2$ be the parametrized curve in the unit 2-sphere S^2 from p (0, 0, 1) to q= (0, 0, −1) defined by $\alpha_\theta(t)=(\cos\theta \sin t, \sin\theta \sin t, \cos t)$. find the parallel transport $P_{\alpha_\theta}(\bar{v})$, where $\bar{v}=(p,1,0,0)$.

23. If X and Y are smooth vector fields on an n-surface S in $\mathbb{R}^{n+1}$ and $\bar{v}$ is a vector tangent to S at $p \in S$, show that $\nabla_{\bar{v}}(X+Y)=\nabla_{\bar{v}}(X)+\nabla_{\bar{v}}(Y)$.

24. If S is the n-sphere $x_1^2+\ldots x_n^2= r^2$ of radius r>0 oriented by the inward unit normal vector field N, prove that $L_p(\bar{v})=\frac{1}{r}\bar{v}$, $\bar{v} \in S_p$.

25. Prove that the curvature at a point p on a curve C in $\mathbb{R}^2$ measures the normal component of acceleration of any unit speed parametrized curve in C passing through p.

26. Find the curvature of the circle C, Where C is the circle $f^{-1}(r^2)$, when $f(x_1,x_2)=(x_1-a)^2+(x_2-b)^2$ and oriented by the outward normal $\nabla f / \|\nabla f\|$.

SECTION- D

Answer any **2** questions out of **4**. Each question carries **6** marks. **(2×6=12)**

27. If U is an open set in $\mathbb{R}^{n+1}$, $f: U \to \mathbb{R}$ is smooth, $p \in U$ is a regular point of f and $c=f(p)$, show that the set of all vectors tangent to $f^{-1}(c)$ at p is equal to $[\nabla f(p)]^\perp$.