K18U 2261 -4-

- 28. If S is the unit circle $x_1^2 + x_2^2 = 1$ and $g: \mathbb{R}^2 \to \mathbb{R}$ is defined by $g(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2$, where $a, b, c \in R$ and $S = f^{-1}(I)$, $f(x_1, x_2) = x_1^2 + x_2^2$ prove that the extreme points of g on S are the eigen vectors of the symmetric matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ and also show that $g(p) = \lambda$, where $p = (x_1, x_2)$ and λ is the eigen value.
- 29. Let $S = f^{-1}(c)$ be an n-surface in \mathbb{R}^{n+1} , where $f: U \to \mathbb{R}$ is such that $\nabla f(g) \neq 0$ for all $q \in S$ and X be a smooth vector field on U whose restriction to S is a tangent vector field on S. If $\alpha: I \to U$ is any integral curve of X such that $\alpha(t_o) \in S$ for some $t_o \in I$, then prove that $\alpha(t) \in S$ for all $t_o \in I$.
- 30. If S is an n-surface \mathbb{R}^{n+1} , oriented by the unit normal vector field \mathbb{N} , $\dot{p} \in S$ and $\mathbb{V} \in Sp$, then prove that for every parametrized curve $\alpha: I \to S$, with $\dot{\alpha}(t_0) = \overline{\nu}$ for some $t_o \in I$, $\ddot{\alpha}(t_0) \cdot \mathbb{N}(p) = L_p(\overline{\nu}) \cdot \overline{\nu}$. (2x6=12)



K18U 2261

V Semester B.Sc. (Hon's) (Mathematics) Degree (Regular) Examination, November 2018 (2016 Admission) BHM 504 : DIFFERENTIAL GEOMETRY

City of Still College, and College, and College,

Time: 3 Hours

Max. Marks: 60

SECTION - A

Answer any 4 questions out of 5 questions. Each question caries one mark. (4×1=4)

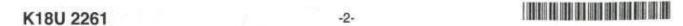
- 1. Define a level set.
- 2. Define the graph of a function $f: U \to \mathbb{R}, U \subseteq \mathbb{R}^{n+1}$.
- 3. Define a vector field \mathbb{X} on $U \subseteq \mathbb{R}^{n+1}$.
- 4. If $f: U \to \mathbb{R}: U \subseteq \mathbb{R}^{n+1}$, define a regular point off.
- 5. If X is a smooth vector field on \mathbb{R}^{n+1} , define the covariant derivative of X.

SECTION - B

Answer any 6 questions out of 9 questions. Each question carries 2 marks. (6x2=12)

- 6. Sketch the vector field \mathbb{X} (p) = (p, X(p)), where X(p) = (1, 0).
- 7. Define an oriented surface. Give an example of an unoriented 2-surface.
- 8. What is meant by a direction at p, where $p \in \mathbb{R}_p^{n+1}$?
- Define a consistent ordered basis for the tangent space Sp. When we say that the ordered basis is inconsistent.

P.T.O.

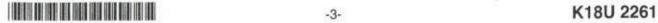


- 10. If S is a surface in \mathbb{R}^{n+1} , $\alpha: 1 \to S$ is a parametrized curve and \mathbb{X} is a smooth vector field and if \mathbb{X} is parallel along α , show that \mathbb{X} has constant length.
- 11. If $f:U\to \mathbb{R},\ U\subseteq \mathbb{R}^{n+1}$ is a smooth function, $\overline{v}\in \mathbb{R}^{n+1}_p$, $p\in U$ and $\alpha:I\to U$ is a parametrized curve, show that the derivative of f with respect to \overline{v} defined by $\nabla_{\overline{v}}(f)=\left(f\circ\alpha\right)'(t_0)$ is independent of α .
- 12. Show that $\nabla_{\overline{v}}(\mathbb{N})$ is tangent to S, where S is an n-surface in \mathbb{R}^{n+1} and \mathbb{N} (p) is the orientation normal direction at p.
- 13. Compute $\nabla_{\overline{v}} f$, where $f(x_1, x_2) = 2x_1^2 + 3x_2^2$ and $\overline{v} = (1, 0, 2, 1)$.
- 14. Find $\nabla_{\overline{v}}(X)$, where $X(x_1, x_2) = (x_1, x_2, x_1, x_2, x_2, x_2)$, $\overline{v} = (1, 0, 0, 1)$.

SECTION - C

Answer any 8 questions out of 12 questions. Each question carries 4 marks. (8×4=32)

- 15. Prove that the gradient of f at $p \in f^{-1}(c)$ is orthogonal to all vectors tangent $f^{-1}(c)$ at p, where $f: U \to \mathbb{R}$, $U \subseteq \mathbb{R}^{n+1}$, $C \in \mathbb{R}$.
- 16. Prove that the n-sphere $x_1^2 + ... + x_{n+1}^2 = 1$ is an n-surface.
- 17. Let S be an n-surface in \mathbb{R}^{n+1} , $S = f^{-1}(c)$, $f : U \to \mathbb{R}$ is such that $\nabla f(q) \neq 0$ for all $q \in S$. Suppose $g : U \to \mathbb{R}$ is a smooth function and $p \in S$ is an extreme point of g on S. Then prove that there exists a real number λ such that $\nabla g(p) = \lambda \nabla f(p)$.
- 18. Let $S \subseteq \mathbb{R}^{n+1}$ be a connected n-surface in \mathbb{R}^{n+1} . Prove that there exist on S exactly two smooth unit normal vector fields \mathbb{N}_1 and \mathbb{N}_2 such that $\mathbb{N}_2(p) = -\mathbb{N}_1(p)$ for all $p \in S$.
- 19. Define a geodesic in n-surface $S \subseteq \mathbb{R}^{n+1}$ and prove that geodesics have constant speed.
- 20. If X and Y are smooth vector fields along the parametrized curve $\alpha: I \to \mathbb{R}^{n+1}$. Prove that X + Y = X + Y.



21. If X and Y are smooth vector fields tangent to S along a parametrized curve $\alpha: I \to S$, show that $(X \cdot Y)' = X' \cdot Y + X \cdot Y'$.

- 22. If S is a 2-surface in \mathbb{R}^3 and $\alpha:I\to S$ is a geodesic in S with $\stackrel{\dot{\alpha}}{}\neq 0$. Then prove that a vector field X tangent to S along α is parallel along α if and only if both ||X|| and the angle between X and $\stackrel{\dot{\alpha}}{}$ are constant along α .
- 23. Let S be an n-surface in \mathbb{R}^{n+1} , let p, q \in S and let α be a piecewise smooth parametrized curve from p to q. Then prove that the parallel transport $P_{\alpha}: S_p \to S_q$ along α is a vector space isomorphism which preserves dot products.
- 24. If S is the n-sphere $x_1^2 + ... + x_{n+1}^2 = r^2$ of radius r > 0 oriented by the inward unit normal vector-field $\mathbb N$, prove that $L_p(\overline{\nu}) = \frac{1}{r} \overline{\nu}$, where $\overline{\nu} \in S_p$,
- 25. Let $C = f^{-1}(c)$, where $f: U \to \mathbb{R}$, be a plane curve in an open set $U \subseteq \mathbb{R}^2$. Define the curvature K(p) of C at p, where $p \in C$, If $\alpha: I \to C$ is a parametrized curve in C with $\dot{\alpha}(t) \neq 0$,, for all $t \in I$, prove that $K(\alpha(t)) = \frac{\ddot{\alpha}(t) \cdot \mathbb{N}(\alpha(t))}{||\dot{\alpha}(t)||^2}$.
- 26. Find the curvature of the circle C, where C in $f^{-1}(r^2)$ and $f(x_1, x_2) = (x_1 a)^2 + (x_2 b)^2$ and oriented by the outward normal $|\nabla f/|| |\nabla f||$.

SECTION - D

Answer any 2 questions out of 4 questions. Each question carries 6 marks.

- 27. Let X be a smooth vector on an open set $U \subseteq \mathbb{R}^{n+1}$ and let $p \in U$. Then prove that there exist an open interval I containing zero and an integral curve $\alpha: I \to U$ of X such that
 - i) $\alpha(0) = p$
 - ii) if $\beta: \tilde{I} \to U$ is any other integral curve of X with $\beta(0) = p$, then $\tilde{I} \subset I$ and $\beta(t) = \alpha(t)$ for all $t \in I$.