28. If S is the unit circle $x_1^2 + x_2^2 = 1$ and g : $\mathbb{R}^2 \rightarrow \mathbb{R}$ is defined by $g(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2$, where a, b, c $\in$ R and S = f^{-1} (I), $f(x_1, x_2) = x_1^2 + x_2^2$ prove that the extreme points of g on S are the eigen vectors of the symmetric matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ and also show that g(p) = λ, where p= (x_1, x_2) and λ is the eigen value.

29. Let S = f^{-1}(c) be an n-surface in $\mathbb{R}^{n+1}$, where f : U $\rightarrow \mathbb{R}$ is such that $\nabla f(g) \neq 0$ for all q $\in$ S and $\mathbb{X}$ be a smooth vector field on U whose restriction to S is a tangent vector field on S. If α : I $\rightarrow$ U is any integral curve of $\mathbb{X}$ such that $\alpha(t_o) \in S$ for some $t_o \in I$, then prove that $\alpha(t) \in S$ for all $t_o \in I$.

30. If S is an n-surface $\mathbb{R}^{n+1}$, oriented by the unit normal vector field $\mathbb{N}$, $\dot{p} \in S$ and $\mathbb{V} \in$ Sp, then prove that for every parametrized curve α : I $\rightarrow$S, with $\dot{\alpha}(t_0) = \bar{v}$ for some $t_o \in$ I, $\ddot{\alpha}(t_0) \cdot \mathbb{N}(p) = L_p(\bar{v}) \cdot \bar{v}$. **(2×6=12)**

Reg. No. : ....

Name : ....

V Semester B.Sc. (Hon's) (Mathematics) Degree (Regular) Examination, November 2018
(2016 Admission)
BHM 504 : DIFFERENTIAL GEOMETRY

Time : 3 Hours Max. Marks : 60

SECTION – A

Answer **any 4** questions out of 5 questions. **Each** question caries **one** mark. **(4×1=4)**

1. Define a level set.

2. Define the graph of a function f : U $\rightarrow \mathbb{R}$, U $\subseteq \mathbb{R}^{n+1}$.

3. Define a vector field $\mathbb{X}$ on U $\subseteq \mathbb{R}^{n+1}$.

4. If f : U $\rightarrow \mathbb{R}$: U $\subseteq \mathbb{R}^{n+1}$, define a regular point off.

5. If $\mathbb{X}$ is a smooth vector field on $\mathbb{R}^{n+1}$, define the covariant derivative of $\mathbb{X}$.

SECTION – B

Answer **any 6** questions out of 9 questions. **Each** question carries **2** marks. **(6×2=12)**

6. Sketch the vector field $\mathbb{X}$ (p) = (p, X(p)), where X(p) = (1, 0).

7. Define an oriented surface. Give an example of an unoriented 2-surface.

8. What is meant by a direction at p, where $p \in \mathbb{R}_p^{n+1}$?

9. Define a consistent ordered basis for the tangent space Sp. When we say that the ordered basis is inconsistent.

10. If S is a surface in $\mathbb{R}^{n+1}$, $\alpha : 1 \to S$ is a parametrized curve and $\mathbb{X}$ is a smooth vector field and if $\mathbb{X}$ is parallel along α, show that $\mathbb{X}$ has constant length.

11. If $f : U \to \mathbb{R}$, $U \subseteq \mathbb{R}^{n+1}$ is a smooth function, $\bar{v} \in \mathbb{R}_p^{n+1}$, $p \in U$ and $\alpha : I \to U$ is a parametrized curve, show that the derivative of f with respect to $\bar{v}$ defined by $\nabla_{\bar{v}}(f) = (f \circ \alpha)'(t_0)$ is independent of α.

12. Show that $\nabla_{\bar{v}}(\mathbb{N})$ is tangent to S, where S is an n-surface in $\mathbb{R}^{n+1}$ and $\mathbb{N}(p)$ is the orientation normal direction at p.

13. Compute $\nabla_{\bar{v}} f$, where $f(x_1, x_2) = 2x_1^2 + 3x_2^2$ and $\bar{v} = (1, 0, 2, 1)$.

14. Find $\nabla_{\bar{v}}(\mathbb{X})$, where $\mathbb{X}(x_1, x_2) = (x_1, x_2, x_1.x_2, x_2^2)$, $\bar{v} = (1, 0, 0, 1)$.

SECTION – C

Answer **any 8** questions out of 12 questions. **Each** question carries **4** marks. **(8×4=32)**

15. Prove that the gradient of f at $p \in f^{-1}(c)$ is orthogonal to all vectors tangent $f^{-1}(c)$ at p, where $f : U \to \mathbb{R}$, $U \subseteq \mathbb{R}^{n+1}$, $C \in \mathbb{R}$.

16. Prove that the n-sphere $x_1^2 + \ldots + x_{n+1}^2 = 1$ is an n-surface.

17. Let S be an n-surface in $\mathbb{R}^{n+1}$, $S = f^{-1}(c)$, $f : U \to \mathbb{R}$ is such that $\nabla f(q) \neq 0$ for all $q \in S$. Suppose $g : U \to \mathbb{R}$ is a smooth function and $p \in S$ is an extreme point of g on S. Then prove that there exists a real number λ such that $\nabla g(p) = \lambda \nabla f(p)$.

18. Let $S \subseteq \mathbb{R}^{n+1}$ be a connected n-surface in $\mathbb{R}^{n+1}$. Prove that there exist on S exactly two smooth unit normal vector fields $\mathbb{N}_1$ and $\mathbb{N}_2$ such that $\mathbb{N}_2(p) = -\mathbb{N}_1(p)$ for all $p \in S$.

19. Define a geodesic in n-surface $S \subseteq \mathbb{R}^{n+1}$ and prove that geodesics have constant speed.

20. If $\mathbb{X}$ and $\mathbb{Y}$ are smooth vector fields along the parametrized curve $\alpha : I \to \mathbb{R}^{n+1}$. Prove that $\dot{\overline{\mathbb{X} + \mathbb{Y}}} = \dot{\mathbb{X}} + \dot{\mathbb{Y}}$.

21. If $\mathbb{X}$ and $\mathbb{Y}$ are smooth vector fields tangent to S along a parametrized curve $\alpha : I \to S$, show that $(\mathbb{X} . \mathbb{Y})' = \mathbb{X}'.\mathbb{Y} + \mathbb{X}.\mathbb{Y}'$.

22. If S is a 2-surface in $\mathbb{R}^3$ and $\alpha : I \to S$ is a geodesic in S with $\dot{\alpha} \neq 0$. Then prove that a vector field $\mathbb{X}$ tangent to S along α is parallel along α if and only if both $\|\mathbb{X}\|$ and the angle between $\mathbb{X}$ and $\dot{\alpha}$ are constant along α.

23. Let S be an n-surface in $\mathbb{R}^{n+1}$, let $p, q \in S$ and let α be a piecewise smooth parametrized curve from p to q. Then prove that the parallel transport $P_\alpha : S_p \to S_q$ along α is a vector space isomorphism which preserves dot products.

24. If S is the n-sphere $x_1^2 + \ldots + x_{n+1}^2 = r^2$ of radius $r > 0$ oriented by the inward unit normal vector-field $\mathbb{N}$, prove that $L_p(\bar{v}) = \frac{1}{r}\bar{v}$, where $\bar{v} \in S_p$.

25. Let $C = f^{-1}(c)$, where $f : U \to \mathbb{R}$, be a plane curve in an open set $U \subseteq \mathbb{R}^2$. Define the curvature K(p) of C at p, where $p \in C$, If $\alpha : I \to C$ is a parametrized curve in C with $\dot{\alpha}(t) \neq 0$, for all $t \in I$, prove that $K(\alpha(t)) = \frac{\ddot{\alpha}(t) \cdot \mathbb{N}(\alpha(t))}{\|\dot{\alpha}(t)\|^2}$.

26. Find the curvature of the circle C, where C in $f^{-1}(r^2)$ and $f(x_1, x_2) = (x_1 - a)^2 + (x_2 - b)^2$ and oriented by the outward normal $\nabla f / \|\nabla f\|$.

SECTION – D

Answer **any 2** questions out of 4 questions. **Each** question carries **6** marks.

27. Let $\mathbb{X}$ be a smooth vector on an open set $U \subseteq \mathbb{R}^{n+1}$ and let $p \in U$. Then prove that there exist an open interval I containing zero and an integral curve $\alpha : I \to U$ of $\mathbb{X}$ such that

    i) $\alpha(0) = p$

    ii) if $\beta : \tilde{I} \to U$ is any other integral curve of $\mathbb{X}$ with $\beta(0) = p$, then $\tilde{I} \subset I$ and $\beta(t) = \alpha(t)$ for all $t \in I$.