31. If a function f is analytic inside and on a positively oriented circle C_R centred at z_0 and with radius R, show that $\left| f^{(n)}(z_0) \right| \le \frac{n!\,M_R}{R^n}$ $(n = 1, 2, 3, \ldots)$, where $|f(z)| \le M_R$ on C_R.

32. If $f(z) = u + iv$ is an analytic function of $z = x + iy$ and ϕ is any function of x and y with differential coefficients of first and second order, prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2.$$

33. Expand in series $f(z) = 1/\left(z^2 - 3z + 2\right)$ in the region :

i) $0 < |z| < 1$ and

ii) $1 < |z| < 2$.

SECTION – D

Answer **any one** question. It carries **10** marks. **(1×10=10)**

34. If $f(z) = u(x, y) + iv(x, y)$ and if $f'(z)$ exists at a point $z_0 = x_0 + iy_0$, show that u and v satisfy the Cauchy-Riemann equations.

35. If f(z) is continuous on a domain D, show that the following statements are equivalent.

i) f(z) has an antiderivative F(z) throughout D.

ii) the integrals of f(z) along contours lying entirely in D and extending from any fixed point z_1 to any fixed point z_2 all have the same value.

iii) the integrals of f(z) around closed contours lying entirely in D all have value zero.

Reg. No. : ....

Name : ....

V Semester B.Sc. Hon's (Mathematics) Degree (Reg./Supple./Improv.) Examination, November 2017

BHM 504 : COMPLEX ANALYSIS – I

(2013-15 Admissions)

Time : 3 Hours Max. Marks : 80

SECTION – A

Answer **all** questions. **Each** question carries **1** mark. **(10×1=10)**

1. Write the function $f(z) = z^3 + z + 1$ in the form $u(x, y) + iv(x, y)$.

2. Give an example of a function f(z) such that its real and imaginary parts satisfy the Cauchy-Riemann equations, but $f'(z)$ does not exist at any non-zero point.

3. Determine the singular points of the function $f(z) = (z^2 + 1)/(z+2)(z^2 + 2z + 2)$.

4. Prove that $|\sinh z|^2 = \sinh^2 x + \sin^2 y$.

5. Evaluate $\int_1^2 \left(\frac{1}{t} - i\right)^2 dt$.

6. By finding an antiderivative, evaluate $\int_i^{1/2} e^{\pi z}\,dz$, where the path is any contour between i and $\frac{1}{2}$.

7. Using Cauchy-Goursat theorem, find $\int_C \exp\left(z^3\right) dz$, where C is any simple closed contour.

8. Define the Maclaurin's series of a function f(z).

9. If $f(z) = \frac{z}{\bar{z}}$, examine whether $\lim\limits_{z \to 0} f(z)$ exist.

10. Find the zeros of sin z.

SECTION – B

Answer **any 10** questions. **Each** question carries **3** marks. **(10×3=30)**

11. If a function f(z) is continuous and non-zero at a point z_0, show that $f(z) \neq 0$ throughout some neighbourhood of that point.

12. Prove that $\frac{d}{dz}(f(z)\,g(z)) = f(z)\,g'(z) + f'(z)\,g(z)$.

13. If a function f(z) = u(x, y) + iv (x, y) is analytic in a domain D, show that its component functions u and v are harmonic in D.

14. Prove that $\sin^{-1} z = -i \log\left[iz + \left(1-z^2\right)^{1/2}\right]$.

15. If C is the positively oriented circle $z = Re^{i\theta}$, $-\pi \leq \theta \leq \pi$, prove that $\int_C \frac{dz}{z} = 2\pi i$.

16. Using Cauchy's integral formula, find $\int_C f(z)\,dz$, where $f(z) = z/\left(9-z^2\right)$ and C is the positively oriented circle | z | = 2.

17. If f is analytic throughout a simply connected domain D, prove that $\int_C f(z)\,dz = 0$.

18. Prove that a function f(z) = u(x, y) + iv (x, y) is analytic in a domain D if and only if v is a harmonic conjugate of u.

19. Find the analytic function f(z) = u + iv of which the real part is $u = e^x(x \cos y - y \sin y)$.

20. Prove that $f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$, $z \neq 0$ and f(0) = 0, satisfy the Cauchy-Riemann equations at the origin.

21. Show that the sequence $z_n = \frac{1}{n^3} + i$, n = 1, 2, ..., converges to i.

22. Prove that the absolute convergence of a series of complex numbers implies the convergence of that series.

23. Expand $f(z) = \left(1+2z^2\right)/\left(z^3+z^5\right)$ into a series involving powers of z for | z | < 1.

24. If a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges when $z = z_1$ ($z_1 \neq z_0$), then prove that it is absolutely convergent at each point z in the open disk $|z - z_0| < R_1$, where $R_1 = |z_1 - z_0|$.

SECTION – C

Answer **any 6** questions. **Each** question carries **5** marks. **(6×5=30)**

25. Show that the line x = a in the 2 plane corresponds to the parabola $v^2 = -4a^2(u - a^2)$ under the mapping $w = z^2$.

26. Let f(z) = u(x, y) + iv(x, y), z = x + iy, $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0$. If $\lim_{z \to z_0} f(z) = w_0$, show that $\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0$.

27. Show that $\int_0^{\pi/4} e^{it}\,dt = 1/\sqrt{2} + i\left(1 - 1/\sqrt{2}\right)$.

28. State and prove Liouville's theorem.

29. Prove that $\int_C \bar{z}\,dz = 4\pi i$, where C is the right hand half $z = 2e^{i\theta}$, $-\pi/2 \leq \theta \leq \pi/2$.

30. If f(z) = u(x, y) + iv(x, y) and its conjugate $\overline{f(z)}$ = u(x, y) – iv (x, y) are both analytic in a given domain D, prove that f(z) is a constant.