Reg. No. : ....

Name : ....

K18U 0319

IV Semester B.Sc. Hon's (Mathematics) Degree (Regular/Supple/Imp.) Examination, May 2018
BHM 401 : REAL ANALYSIS – II
(2013 – 2015 Admns.)

Time : 3 Hours Marks : 80

Answer **all** the **10** questions. **(10×1=10)**

1. If I = [0, 4], calculate the norm of the partition $P_1 = (0, 1, 2, 4)$.
2. Define a tagged partition of the closed interval [a, b].
3. State the Cauchy criterion for the Riemann integrability of a function $f : [a, b] \to \mathbb{R}$.
4. Define the n^{th} Simpson approximation.
5. State the Trapezoidal rule.
6. Find $\lim \left(\frac{x^2 + nx}{n}\right)$.
7. If $f : D \to \mathbb{R}$, where $D \leq \mathbb{R}^n$, define the partial derivative of f with respect to x_i at $X = (x_1, x_2, \ldots, x_n) \in D$.
8. Define the derivative of a function $f : \mathbb{R}^2 \to \mathbb{R}$.
9. If a function $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (x_0, y_0), what is the maximum value of $D_u f(x_0, y_0)$.
10. If $x = t^2$, $y = t^3$ and $z = xy$, find $\frac{dz}{dt}$.

Answer **any 10** short answer questions out of 14. **(10×3=30)**

11. Show that every constant function on [a, b] is in $\Re$ [a, b].
12. If $f \in \Re$ [a, b] and $|f(x)| \leq M$ for all $x \in [a, b]$, show that $\left|\int_a^b f\right| \leq M(b-a)$.

P.T.O.

13. If J is a subinternal of [a, b] having end points c < d and if $\varphi_j(x) = 1$ for $x \in J$ and $\varphi_j(x) = 0$ elsewhere in [a, b], show that $\varphi_j \in \Re[a, b]$ and $\int_a^b \varphi_j = d - c$.

14. If $f \in \Re[a, b]$ and if $[c, d] \le (a, b]$, prove that the restriction of f to [c, d] is in $\Re[c, d]$.

15. If $K(x) = x^2 \cos\left(\frac{1}{x^2}\right)$ for $x \in (0, 1]$ and K(0) = 0, show that the fundamental theorem of calculus does not apply to K^1.

16. Prove that the set Q_1 of rational numbers in [0, 1] is a null set.

17. If f and g belong to $\Re[a, b]$, show that fg belongs to $\Re[a, b]$.

18. Prove that the indefinite integral F defined by $F(z) = \int_a^z f$, for $z \in [a, b]$, is continuous on [a, b].

19. Find $D_{\bar{u}} f(1, 1)$, where $f(x, y) = x^2 + xy + 2$ and $\bar{u} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

20. If $f(x, y) = x^2 - y^2 + 2y$, verify whether $f_{112} = f_{121} = f_{211}$.

21. If $f : \mathbb{R}^n \to \mathbb{R}$ is a differentiable function on an open set D, then prove that at each point x in D at which the gradient of f does not vanish the maximum rate of change of f is in the direction $\nabla f(x)$ and the magnitude of the rate of increase in this direction is $|\nabla f(x)|$.

22. If $z = x^2 + y^3$, where $x = st$, $y = e^{st}$ find $\frac{\partial z}{\partial s}$.

23. Show that $u = e^x \cos y$ and $v = e^x \sin y$ satisfy the Cauchy-Riemann equations.

24. If $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (x_0, y_0), show that $D_{\bar{u}} f(x_0, y_0) = |\nabla f(x_0, y_0)|$.

Answer **any 6** short essay questions out of 9. **(6×5=30)**

25. If $f \in \Re$ [a, b], show that the value of the integral in uniquely determined.

26. If $f, g \in \Re$ [a, b] and $f(x) \le g(x)$ for all $x \in [a, b]$, show that $\int_a^b f \le \int_a^b g$.

27. Prove that the Thomae's function $h : [0, 1] \to \Re$ defined by $h(x) = 0$ if $x \in [0, 1]$ is rational, $h(0) = 1$ and $h(x) = \frac{1}{n}$ if $x \in [0, 1]$ is the rational number $x = \frac{m}{n}$, $m, n \in \mathbb{N}$, $(m, n) = 1$, is Riemann integrable on [0, 1].

28. If $f \in \Re$ [a, b] and α, β, γ are any real numbers in [a, b] and if any of the two integrals $\int_\alpha^\beta f$, $\int_\alpha^\gamma f$, $\int_\gamma^\beta f$, exist, show that the third also exists.

29. Suppose that the functions F and G are differentiable on [a, b] and $f = F'$ and $g = G'$ belong to $\Re$ [a, b]. Prove that $\int_a^b fG = FG\Big|_a^b - \int_a^b Fg$.

30. Given that (f_n) is a sequence of continuous functions defined on a set $A \le \mathbb{R}$ and (f_n) converges uniformly to a function f. Then show that f is continuous.

31. If R is the radius of convergence of $\Sigma a_n x^n$ and K is a closed bounded interval contained in the interval of convergence (–R, R), prove that the power series converges uniformly.

32. Let f be defined in a neighbourhood of $(x_0, y_0) \in \mathbb{R}^2$. Suppose f has partial derivatives f_1, f_2, f_{12} and f_{21} in this neighbourhood and that the mixed partials f_{12} and f_{21} are continuous at (x_0, y_0). Show that $f_{12}(x_0, y_0) = f_{21}(x_0, y_0)$.

33. If f is differentiable at $(x_0, y_0) \in \mathbb{R}^2$ and if $\bar{u} = (u_1, u_2)$ is any unit vector, show that $D_{\bar{u}} f(x_0, y_0)$ exists and $D_{\bar{u}} f(x_0, y_0) = f_1(x_0, y_0) u_1 + f_2(x_0, y_0) u_2$.

Answer **any one** essay question out of 2. **(1×10=10)**

34. State and prove the Cauchy criterion for the Riemann integrability of a function f.

35. Let f be continuous on the rectangle $R = [a, b] \times [c, d]$ and let $F(y) = \int_a^b f(x, y)dx$ for each $y \in [c, d]$. If the partial derivative $\frac{\partial f}{\partial y}$ exists and is continuous on R, prove that F is differentiable on R and $F'(y) = \int_a^b \frac{\partial}{\partial y} f(x, y)dx$.

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