29. If f : [a, b] $\to \mathbb{R}$, show that $f \in \Re[a, b]$ if and only if for every $\varepsilon > 0$ there exist functions α_ε and ω_ε in $\Re[a, b]$ with $\alpha_\varepsilon(x) \le f(x) \le \omega_\varepsilon(x)$ for all $x \in [a, b]$ and such that $\int_a^b (\omega_\varepsilon - \alpha_\varepsilon) < \varepsilon$.

30. If (X, d) is a metric space, prove the following :

i) ϕ and X are open sets in (X, d).

ii) the union of any finite, countable or uncountable family of open sets is open.

iii) the intersection of any finite family of open sets is open.

Reg. No. : ....

Name : ....

IV Semester B.Sc. Hon's (Mathematics) Degree (Regular/Supple./ Improve.) Examination, May 2018

BHM 401 : ADVANCED REAL ANALYSIS AND METRIC SPACES

(2016 Admission Onwards)

Time : 3 Hours Max. Marks : 60

SECTION – A

Answer any 4 questions out of **5** questions. **Each** question carries **1** mark. **(4×1=4)**

1. If I = [0, 4], find the norm of the partition P = (0, 1, 1.5, 2, 3.4 , 4)
2. Define a Riemann integrable function.
3. Find $\lim\left(\frac{x}{n}\right)$ for $x \in \mathbb{R}$.
4. Define uniform convergence of a sequence of functions on $A \subseteq \mathbb{R}$.
5. Define the discrete metric on a set X.

SECTION – B

Answer **any 6** questions out of **9** questions. **Each** questions carries **2** marks. **(6×2=12)**

6. Prove that every constant function on [a, b] is in $\Re[a, b]$.
7. Let $g : [0,3] \to \mathbb{R}$ be defined by g (x) = 2 for $0 \le x \le 1$ and g (x) = 3 for $1 < x \le 3$. Show that $\int_0^3 g = 8$.

8. If J is a subinterval of [a, b] having end points c < d and $\varphi(x) = 1$ for $x \in J$ and $\varphi_j(x) = 0$ elsewhere in [a, b], show that $\int_a^b \varphi_j = d - c$.

9. Show that any step function is Riemann integrable.

10. If $f \in \Re[a, b]$ and if $[c, d] \subseteq [a, b]$, show that the restriction of f to [c, d] is in $\Re[c, d]$.

11. If f is continuous on [a, b], $f(x) \geq 0$ for all $x \in [a, b]$ and $\int_a^b f = 0$, show that $f(x) = 0$ for all $x \in [a, b]$.

12. Prove that the function $f(x) = \frac{x}{1+x}$ is monotonically increasing.

13. Give an example of a Cauchy sequence in a metric space X such that it does not converge to a point of the space.

14. If A is a subset of a metric space (X, d), prove that A° is an open subset of A that contains every open subset of A.

SECTION – C

Answer **any 8** questions out of **12** questions. **Each** questions carries **4** marks. **(8×4=32)**

15. If g is Riemann integrable on [a, b] and if f (x) = g (x) except for a finite number of points in [a, b], show that f is Riemann integrable and $\int_a^b f = \int_a^b g$.

16. If $f \in R[a, b]$, show that the value of the integral is uniquely determined.

17. If f and g are in R [a, b] and $f(x) \leq g(x)$ for all $x \in [a, b]$, show that $\int_a^b f \leq \int_a^b g$.

18. Prove that the Dirichlet function, defined by f (x) = 1 if $x \in [0, 1]$ is rational and f (x) = 0 if $x \in [0, 1]$ is irrational, is not Riemann integrable.

19. If $f : [a, b] \to \mathbb{R}$ is continuous on [a, b], show that $f \in \Re[a, b]$.

20. If $f : [a, b] \to \mathbb{R}$ is monotone on [a, b], show that $f \in \Re[a, b]$.

21. If $f \in \Re[a, b]$, prove that the indefinite integral of f, defined by $F(z) = \int_a^z f$, is continuous on [a, b].

22. Prove that a sequence (f_n) of bounded functions on $A \subseteq \mathbb{R}$ converges uniformly on A if and only if $\|f_n - f\| \to 0$.

23. State and prove the Cauchy-Hadamard theorem.

24. State and prove the Holder's inequality.

25. If a Cauchy sequence of points in a metric space (X, d) contains a convergent subsequence, prove that the sequence converge to the same limit as the subseqeunce.

26. Prove that a subset G in a metric space (X, d) is open if and only if it is the union of all open balls contained in G.

SECTION – D

Answer **any 2** questions out of **4** questions. **Each** carries **6** marks. **(2×6=12)**

27. Let h (x) = x for $x \in [0, 1]$, show that $h \in \Re[0, 1]$.

28. State and prove the Cauchy criterion for the Riemann integrability of a function $f : [a, b] \to \mathbb{R}$