K20U 0217

Reg. No. : ....

Name : ....

IV Semester B.Sc. Hon's (Mathematics) Degree (Reg./Supple./Improv.) Examination, April 2020
(2016 Admission Onwards)
BHM 401 : ADVANCED REAL ANALYSIS AND METRIC SPACES

Time : 3 Hours Max. Marks : 60

SECTION – A

Answer **any 4** questions out of 5 questions. **Each** question carries **1** mark. **(4×1=4)**

1. If $f(x) = x^2$ for $x \in [0, 4]$, calculate the Riemann sum corresponding to the partition $P = (0, 2, 3, 4)$ with tags at the left end points of the subintervals.

2. Give an example of a function $f : [a, b] \to \mathbb{R}$ that is in $\mathcal{R}[c, b]$ for every $c \in (a, b)$ but which is not in $\mathcal{R}[a, b]$.

3. Find $\lim (x^2 + nx)/n$ for $x \in \mathbb{R}$.

4. State the Cauchy criterion for uniform convergence of a sequence of functions.

5. Obtain the derived set F', where $F = \{1, ½, ⅓, ...\}$.

SECTION – B

Answer **any 6** questions out of 9 questions. **Each** question carries **2** marks. **(6×2=12)**

6. If $G(x) = \frac{1}{n}$ for $x = \frac{1}{n} (n \in \mathbb{N})$ and $G(x) = 0$ elsewhere on [0, 1], show that $\int_0^1 G = 0$.

7. If f and g are in $\mathcal{R}[a, b]$, show that $\int_a^b kf = k\int_a^b f$, where $k \in \mathbb{R}$.

8. If $\varphi : [a, b] \to \mathbb{R}$ is a step function, show that $\varphi \in \mathcal{R}[a, b]$.

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9. If $f \in \mathcal{R}[a, b]$ and if α, β, γ are any numbers in [a, b], show that $\int_\alpha^\beta f = \int_\alpha^\gamma f + \int_\gamma^\beta f$ in the sense that the existence of any two of these integrals implies the existence of the third and equality holds.

10. If $f \in \mathcal{R}[a, b]$ and if $[c, d] \subseteq [a, b]$ show that the restriction of f to [c, d] is in $\mathcal{R}[c, d]$.

11. Consider the function f defined by $f(x) = x + 1$ for $x \in [0, 1]$ rational and $f(x) = 0$ for $x \in [0,1]$ irrational. Show that f is not Riemann integrable.

12. If (X, d) is a metric space and $d' : X \times X \to \mathbb{R}$ is defined by $d'(x, y) = d(x, y)/1 + d(x, y)$, prove that d′ is a metric on X.

13. In any metric space (X, d), prove that every open ball is an open set.

14. Give an example for $(A \cup B)^\circ \neq A^\circ \cup B^\circ$, where $A \subseteq \mathbb{R}$, $B \subseteq \mathbb{R}$.

SECTION – C

Answer **any 8** questions out of 12 questions. **4** marks **each**. **(8×4=32)**

15. If $f \in \mathcal{R}[a, b]$, show that the value of the integral is unique.

16. If f and g are in $\mathcal{R}[a, b]$, show that f + g is in $\mathcal{R}[a, b]$ and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

17. Define the Thomae's function $h : [0, 1] \to \mathbb{R}$ and prove that it is in $\mathcal{R}[0, 1]$ with integral zero.

18. If f and g are in $\mathcal{R}[a, b]$ and $f(x) \leq g(x)$ for all $x \in [a, b]$, show that $\int_a^b f \leq \int_a^b g$.

19. If $f : [a, b] \to \mathbb{R}$ is monotone on [a, b], show that $f \in \mathcal{R}[a, b]$.

20. If E is a finite set in [a, b] and there exist functions $f, F : [a, b] \to \mathbb{R}$ such that (i) F is continuous on [a, b] (ii) $F'(x) = f(x)$ for all $x \in [a, b] \setminus E$ (iii) $f \in \mathcal{R}[a, b]$, show that $\int_a^b f = F(b) - F(a)$.

21. Evaluate $\int_1^4 \frac{\sin\sqrt{t}}{\sqrt{t}}\, dt$ using the Substitution theorem.

22. Let (f_n) be a sequence of continuous functions on a set $A \subseteq \mathbb{R}$ and suppose that (f_n) converges uniformly on A to a function $f : A \to \mathbb{R}$. Show that f is continuous on A.

23. If R is the radius of convergence of $\sum a_n x^n$ and K is a closed, bounded interval contained in the interval of convergence. (–R, R), prove that the power series converges uniformly on K.

24. State and prove Minkowski's inequality.

25. Prove that every convergent sequence in a metric space is a Cauchy sequence.

26. Let (X, d) be a metric space and F be a subset of X. Prove that F is closed in X if and only if F^c is open in X.

SECTION – D

Answer **any 2** questions out of 4. **Each** carries **6** marks. **(2×6=12)**

27. If $f \in \mathcal{R}[a, b]$, show that f is bounded on [a, b].

28. Let $f : [a, b] \to \mathbb{R}$ and let $c \in (a, b)$. Prove that $f \in \mathcal{R}[a, b]$ if and only if its restrictions to [a, c] and [c, b] are both Riemann integrable.

29. Prove that the metric space (X, d), where X denote the space of all sequences $x = (x_1, x_2, ...)$ of real numbers for which $\left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} < \infty$ $(p \geq 1)$ and d is the metric given by $d_p(x, y) = \left(\sum_{k=1}^{\infty} |x_k - y_k|^p\right)^{1/p}$, is a complete metric space.

30. Prove that each nonempty open subset of $\mathbb{R}$ is the union of a countable family of distinct open intervals.