Reg. No. : ....

Name : ....

K16U 1336

IV Semester B.Sc. Hon's (Mathematics) Degree (Reg./Supple./Improve.) Examination, May 2016
BHM 402 : ABSTRACT ALGEBRA – II

Time : 3 Hours Max. Marks : 80

Answer **all** the **10** questions. **(10×1=10)**

1. Define the direct product of the groups $G_1, G_2, ..., G_n$.
2. Give the order of (1, 0) in $\mathbb{Z}_2 \times \mathbb{Z}_4$.
3. Find all proper nontrivial subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$.
4. Find the order of the factor group $\mathbb{Z}_6 / \langle 3 \rangle$.
5. Define the centre Z (G) of a group G.
6. Give an example of a normal series of $\mathbb{Z}$.
7. Define a subnormal series of a group G.
8. If G is a group and X is a G-set, define an orbit in X under G.
9. Define the class equation of a group G.
10. Define a free group and give an example .

Answer **any 10** short answer questions out of **14**. **(10×3=30)**

11. Examine whether is $\mathbb{Z}_3 \times \mathbb{Z}_3$ is cyclic.
12. Prove that the finite indecomposable abelian groups are exactly the cyclic groups with order a power of a prime.
13. If H is a normal subgroup of a group G, prove that the cosets of H in G form a group under the binary operation (aH) (bH) = (ab) H.
14. Prove that a factor group of a cyclic group is cyclic.

P.T.O.

15. If G has a composition series and if N is a proper normal subgroup of G, show that there exists a composition series containing N.

16. If $\phi : \mathbb{Z}_{12} \to \mathbb{Z}_3$ is the homomorphism such that $\phi(1) = 2$, find
 i) Kernel K of ϕ and
 ii) List the cosets in $\mathbb{Z}_{12}/K$ showing the elements in each case.

17. Find any three composition series of $\mathbb{Z}_{48}$.

18. If X is a G-set and $\sigma_g : X \to X$ defined by $\sigma_g(x) = gx$, $x \in X$, is a permutation of X, show that $\phi : G \to S_x$ defined by $\phi(g) = \sigma_g$ is a homomorphism.

19. If X is a G-set, show that $G_x = \{g \in G / gx = x\}$ is a subgroup of G, for each $x \in X$.

20. Suppose G is a group of order p^n and X is a finite G-set. Show that $|X| \equiv |X_G|$ (mod p).

21. Prove that every group of prime-power order is solvable.

22. Prove that every group of order 15 is cyclic.

23. Prove that $\mathbb{Z}_n$ is not free abelian.

24. Show that $(x, y : y^2 x = y, y x^2 y = x)$ is a presentation of the trivial group of one element.

Answer **any 6** short essay questions out of **9**. **(6×5=30)**

25. Prove that $\mathbb{Z}_2 \times \mathbb{Z}_3$ is a cyclic group.

26. If m divides the order of a finite abelian group G, prove that G has a subgroup of order m.

27. Suppose G is a group and H is a subgroup of G. Prove that the following conditions are equivalent
 i) $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$
 ii) $gHg^{-1} = H$ for all $g \in G$
 iii) $gH = Hg$ for all $g \in G$.

28. If N is a normal subgroup of G and H is any subgroup of G, show that $H \vee N = HN = NH$.

29. If X is a G-set and $x \in X$, show that $|Gx| = (G{:}G_x)$.

30. If H is a p-subgroup of a finite group G, show that $(N[H] : H) \equiv (G : H) \pmod p$.

31. Suppose G is a group containing normal subgroups H and K such that $H \cap K = \{e\}$ and $H \vee K = G$. Show that G is isomorphic to $H \times K$.

32. If H and K are finite subgroups of a group G, show that $|HK| = \dfrac{(|H|)(|K|)}{|H \cap K|}$.

33. If $X = \{x_1, x_2, ..., x_r\}$ is a basis for a free abelian group G and $t \in \mathbb{Z}$, then for $i \neq j$, show that the set $Y = \{x_1, ..., x_{j-i}, x_j + tx_i, x_j, x_{j+1} ..., x_r\}$ is also a basis for G.

Answer **any one** essay question out of **2**. **(1×10=10)**

34. If G is a group, prove that the set of all commutators $a b a^{-1} b^{-1}$ for $a, b \in G$ generates a subgroup C of G. Prove also that C is normal subgroup of G, and if N is a normal subgroup of G, then G/N is abelian if and only if $C \leq N$.

35. State and prove the first Sylow theorem.