32. For a prime number p, prove that every group G of order p^2 is abelian.

33. If G is a finitely generated abelian group with generating set $\{a_1, a_2, ..., a_n\}$, prove that $\phi : \mathbb{Z} \times \mathbb{Z} \times .... \times \mathbb{Z} \to G$ defined by

$\phi(h_1, h_2,..., h_n) = h_1a_1 + h_2a_2 + ... + h_na_n$ is a homomorphism.

Answer **any one** essay question out of 2 : **(1×10=10)**

34. Prove that $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic and is isomorphic to $\mathbb{Z}_{mn}$ if and only if m and n are relatively prime.

35. If G is a finite group and p divides |G|, p is a prime, show that G has an element of order p and consequently, a subgroup of order p.

Reg. No. : ....

Name : ....

IV Semester B.Sc. (Hon's) Degree (Mathematics (Regular)) Examination, May 2015
BHM 402 : ABSTRACT ALGEBRA – II

Time : 3 Hours Max. Marks : 80

Answer **all** the **ten** questions : **(10×1=10)**

1. Define the Cartesian product of the sets $S_1, S_2, ..., S_n$.

2. Define a decomposable group.

3. Find the order of the element (2, 6) in $\mathbb{Z}_4 \times \mathbb{Z}_{12}$.

4. State the fundamental homomorphism theorem.

5. Define a simple group and give an example.

6. Give an example of a normal series of $\mathbb{Z}$.

7. State the Schreier Theorem.

8. Define an action of a group G on a set X.

9. Find the reduced form of the word $a^2b^{-1}b^3a^3c^{-1}c^4b^{-2}$.

10. Verify whether {(2, 1), (4, 1)} is a basis for $\mathbb{Z} \times \mathbb{Z}$.

Answer **any 10** short answer questions out of 14 : **(10×3=30)**

11. If a_i is of order r_i in the group G_i, $i = 1, 2, ..., n$, prove that the order of $(a_1, a_2, ..., a_n)$ in $\prod_{i=1}^{n} G_i$ is the least common multiple of all the r_i.

12. Find the order of (8, 4, 10) in the group $\mathbb{Z}_{12} \times \mathbb{Z}_{60} \times \mathbb{Z}_{24}$.

13. If m is a square free integer, prove that every abelian group of order m is cyclic.

14. If H is a normal subgroup of a group G, prove that $\gamma = G \to G/H$ given by $\gamma(x) = xH$ is a homomorphism with kernel H.

15. If G has a composition series and if N is a proper normal subgroup of G, show that there exists a composition series containing N.

16. If $\phi : \mathbb{Z}_{12} \to \mathbb{Z}_3$ is the homomorphism such that $\phi(1) = 2$, find (a) the kernel of ϕ and (b) list the cosets in $\mathbb{Z}_{12}/K$ showing the elements in each coset.

17. Give isomorphic refinements of the two series $\{0\} < 10\mathbb{Z} < \mathbb{Z}$ and $\{0\} < 25\mathbb{Z} < \mathbb{Z}$.

18. If G is a group and X is a G-set, show that the function $\sigma_g : X \to X$ defined by $\sigma_g(x) = gx$, for $x \in X$, is a permutation of X.

19. If G is a group and X is a G-set, show that $G_x = \{g \in G / gx = x\}$ is a subgroup of G, for each $x \in X$.

20. Show that the relation '~' defined on a G-set X given by $x_1 \sim x_2$ if and only if there exist $g \in G$ such that $gx_1 = x_2$, for $x_1, x_2 \in X$, is an equivalence relation.

21. If H is a finite subgroup of a group G and if $ghg^{-1} \in H$ for all $h \in H$ and $g \in G$, show that $g \in N[H]$.

22. If p and q are distinct primes with $p < q$, show that every group G of order pq has a single subgroup of order q, which is normal in G.

23. Prove that $\mathbb{Z}_n$ is not free abelian.

24. Prove that every group G' is a homomorphic image of a free group G.

Answer **any 6** short essay questions out of 9 : **(6×5=30)**

25. If $G_1, G_2, ..., G_n$ are groups and if for $(a_1, a_2, ..., a_n), (b_1, b_2, ..., b_n)$ in $\prod_{i=1}^{n} G_i$ their product is defined as $(a_1b_1, a_2b_2, ..., a_nb_n)$, show that $\prod_{i=1}^{n} G_i$ is a group.

26. If m divides the order of a finite abelian group G, prove that G has a subgroup of order m.

27. If G is a group and H is a subgroup of G, prove that (aH) (bH) = (ab)H is well defined if and only if H is a normal subgroup of G.

28. If N is a normal subgroup of a group G and $\gamma : G \to G/N$ is a canonical homomorphism, prove that the map ϕ from the set of normal subgroups of G containing N to the set of normal subgroups of G/N given by $\phi(L) = \gamma(L)$ is one-to-one and onto.

29. If X is a G-set, show that $|Gx| = (G : G_x)$, $x \in X$.

30. If G is a finite group, X is a finite G-set and r is the number of orbits in X, show that $r \cdot |G| = \sum_{g \in G} |X_g|$.

31. If P_1 and P_2 are Sylow p-subgroups of a finite group G, show that P_1 and P_2 are conjugate subgroups of G.