K16U 1341

Reg. No. : ……………………………

Name : ……………………………

II Semester B.Sc. Hon's (Mathematics) Degree (Reg./Supple./Improve.) Examination, May 2016
BHM 202 : ABSTRACT ALGEBRA – I

Time : 3 Hours　　　　Max. Marks : 80

Answer **all** the **10** questions. **(10×1=10)**

1. Examine whether the usual addition + on the set of all real numbers induce a binary operation on the set of all non-zero reals.

2. If an operation $*$ on the set of all rationals Q is defined by $a*b = a/b$, check whether $*$ is a binary operation on Q.

3. Is the Klein - 4 group V = {e, a, b, c} – cyclic ? Obtain the proper cyclic subgroups of V, if any.

4. Prove that the subgroups of $\mathbb{Z}$ under-addition are precisely the groups n $\mathbb{Z}$ under addition.

5. If (Q, +) is the additive group of rational, examine whether $\mathbb{Z}$ is a subgroup Q^+.

6. If $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 6 & 5 & 4 & 1 \end{pmatrix}$ and
$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 5 & 6 & 2 & 4 \end{pmatrix}$, find $\alpha\beta$ and $(\alpha\beta)^+$.

7. Examine whether the product of the cycles (1, 4, 5, 6) and (2, 1, 5) in S_6 is again a cycle.

8. Show that $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 3 & 2 & 6 & 5 \end{pmatrix}$ in S_6 is an odd permutation.

P.T.O.

9. Show that the map $\phi : \mathbb{Z} \to 2\mathbb{Z}$ defined by $\phi(x) = 2x$ for $x \in \mathbb{Z}$ is not a ring homomorphism.

10. If p is a prime, show that $\mathbb{Z}_p$ has no zero divisors.

Answer **any 10** short answer questions out of **14**. **(10×3=30)**

11. Show that the binary structures $(\mathbb{R}, +)$ and $(\mathbb{R}, .)$ are isomorphic, where + is the usual addition, '.' is the multiplication on $\mathbb{R}$, the set of all reals.

12. Show that the set of all invertible n × n matrices with real entries is a group with respect to matrix multiplication.

13. If $(G, *)$ is a group, show that the linear equations $a * x = b$ and $y * a = b$ have unique solutions in G.

14. If $(a * b)^2 = a^2 * b^2$ for 'a' and 'b' in a group G, show that $a * b = b * a$, where $a^2 = a * a$.

15. Show that every cyclic group is abelian.

16. State and prove the division algorithm for $\mathbb{Z}$.

17. Prove that a non-empty subset H of a group G is a subgroup of G if and only if $a, b \in H$ implies $ab^{+} \in H$.

18. Find all subgroups of $\mathbb{Z}_{18}$.

19. Prove that any permutation of a finite set of at least two elements is a product of transpositions.

20. Show that the number of even permutations in S_n is the same as the number of odd permutations.

21. Prove that every group of prime order is cyclic.

22. Prove that a homomorphism ϕ from a group G to another Group G' is one-to-one if and only if Ker $\phi = \{e\}$.

23. Prove that every field F is an integral domain.

24. In the ring $\mathbb{Z}_n$, prove that the divisors of zero are precisely those non-zero elements that are relatively prime ton.

Answer **any 6** short essay questions **out** of **9**. **(6×5=30)**

25. Show that the binary structures $(\mathbb{Z}, +)$ and $(2\mathbb{Z}, +)$ are isomorphic, where $2\mathbb{Z} = \{2n : n \in \mathbb{Z}\}$.

26. If G is a group with binary operation $*$, prove that the left and right cancellation laws hold in G.

27. If $(G, *)$ is a group, show that the identity element and inverse of each element are unique in G.

28. If G is a group and $a \in G$, show that $H = \{a^n : n \in \mathbb{Z}\}$ is a subgroup of G.

29. Show that every group is isomorphic to a group of permutations.

30. If G and G^1 are groups and $\phi : G \to G^1$ is a one-to-one function such that $\phi(xy) = \phi(x)\,\phi(y)$ for all $x, y \in G$, show that $\phi(G)$ is a subgroup of G. Show also that ϕ provides as isomorphism of G with $\phi[G]$.

31. If $\phi : G \to G^1$ is a homomorphism of a group G into a group G^1, prove the following.
   i) If e is the identity element of G, then $\phi(e)$ is the identity element of G^1
   ii) $\phi(a^{-1}) = \phi(a)^{-1}$, where $a \in G$.
   iii) If H is a subgroup of G, then $\phi[H]$ is a subgroups of G^1.

32. Prove that every finite integral domain is a field.

33. Let R be aring with unity. If $n.1 \neq 0$ for all $n \in \mathbb{Z}^+$, show that R has characteristic zero. If $n.1 = 0$ for some $n \in \mathbb{Z}^+$, then show that the smallest such integer is the characteristic of R.

Answer **any one** essay question **out** of **2**. **(1×10=10)**

34. If G is a cyclic group with generator 'a' and if order of G is finite, show that G is isomorphic to $(\mathbb{Z}, +)$. If G has finite order n, then show that G is isomorphic to $\mathbb{Z}_n$, under addition modulo n.

35. Prove that every group is isomorphic to a group of permutations.
